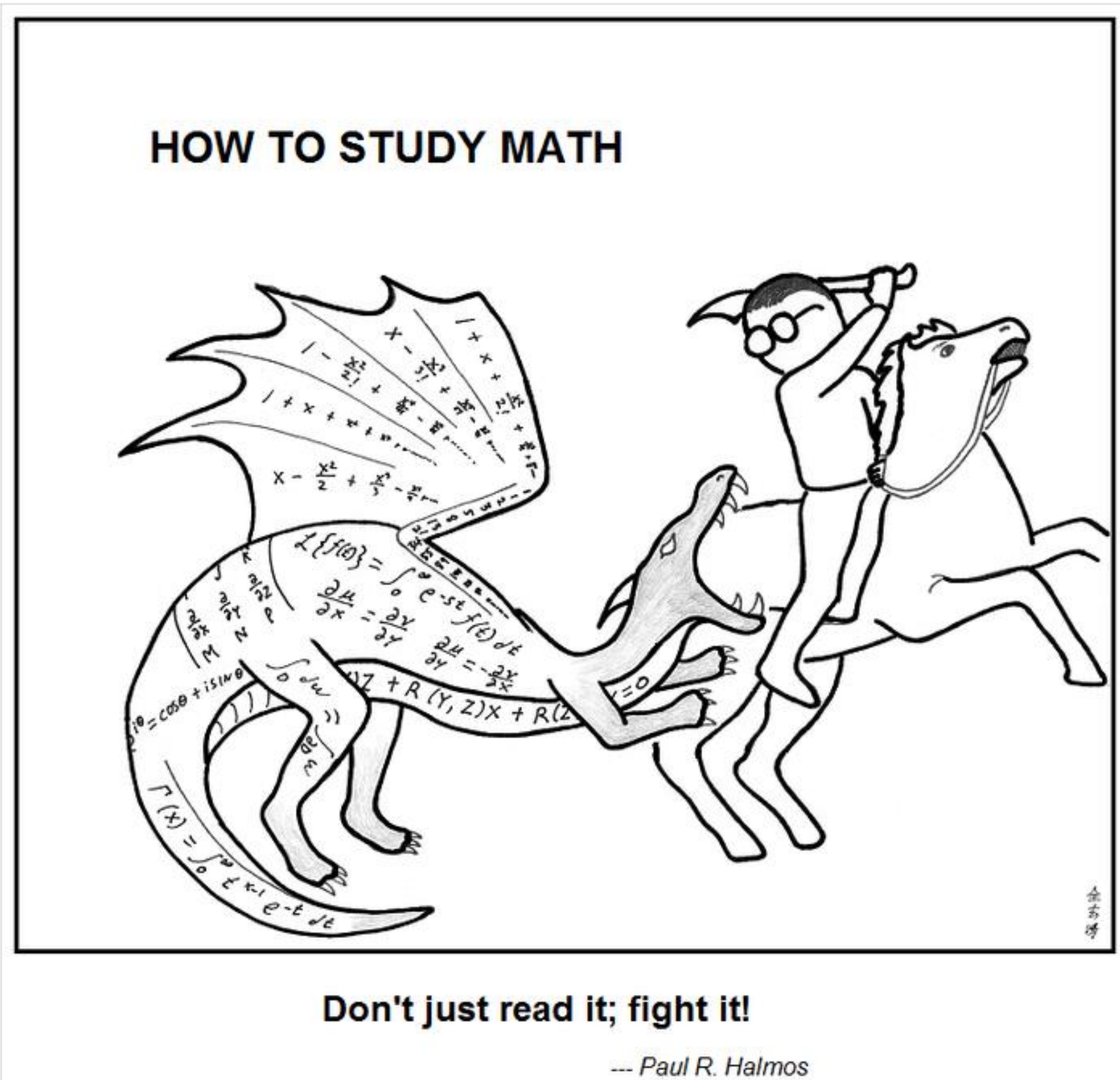
MATH 351 PRACTICE TEST III 14TH NOVEMBER 2018

ABSTRUSE GOOSE



PART I *Definitions and statements of theorems*

PART II (*Each of the following 14 assertions is false. Give an explicit counter-example to illustrate this.*

1. If H: $(0, 1) \to \mathbf{R}$ is continuous, then H is unbounded.
2. If G: $\mathbf{R} \to \mathbf{R}$ is discontinuous at every $x \in \mathbf{R}$, then G^2 is discontinuous at every $x \in \mathbf{R}$.
3. There does not exist a function g: $\mathbf{R} \to \mathbf{R}$ that is discontinuous *only* at every $x \in \mathbf{N}$.
4. There does not exist a function f: $\mathbf{R} \to \mathbf{R}$ that is continuous *only* at every $x \in \mathbf{N}$.

5. If G: [0, 1] $\rightarrow$ **R** is continuous then the set S = {x $\in$ [0, 1]: G(x) = 0} is either uncountable or finite (possibly empty).

6. Let I = (0, 1). There does not exist a function g: I $\rightarrow$ **R** such that g(I) is a compact interval of positive length.

7. If two functions, f: **R** $\rightarrow$ **R** and g: **R** $\rightarrow$ **R** satisfy the conditions that

$$\lim_{x\to 3} f(x) = 4, \ \lim_{x\to 4} g(x) = 5$$

and f is continuous at x = 3,

$$\text{then } \lim_{x\to 3} g \circ f(x) = 5.$$

8. If F: [0, 1] $\rightarrow$ **R** is continuous and $\{x_n\} \subset [0, 1]$ is a sequence for which $F(x_n)$ converges then $\{x_n\}$ must converge.

9. If f: [a, b] $\rightarrow$ **R** satisfies the Intermediate Value Property on [a, b], then *f* is continuous on [a, b].

10. Let F: **R**$\rightarrow$ **R** be continuous and let A, B $\subseteq$ **R**. Then F(A $\cap$ B) = F(A) $\cap$ F(B).

11. If *S* and *T* are sequentially compact subsets of R, then $A \setminus B = \{x \in S \mid x \notin T\}$ is sequentially compact.

12. Let G: (0, 1) $\rightarrow$ **R** be a continuous function and let $\{a_n\} \subset (0, 1)$ be a Cauchy sequence. Then $\{G(a_n)\}$ is a Cauchy sequence.

13. Let h: [0,1] $\rightarrow$ **R** be a function that achieves a maximum value on the interval [0, 1]. Then the function $F(x) = (h(x))^2$ also achieves a maximum value on the interval [0, 1].

14. Let f: **R** $\rightarrow$ **R** and g: **R** $\rightarrow$ **R** each have a jump discontinuity at x = 5. Then f + g has a jump discontinuity at x = 5.

PART III

Instructions: Select any 3 of the following 4 problems. You may answer all 4 to obtain extra credit.

1. Let I = [a, b] be a compact interval. Prove that *I* is sequentially compact.

2. Using *only the definition of limit*, prove that $\lim_{x\to 0} \frac{5-x}{x^2+1} = 5.$

3. Let G: (a, b) $\rightarrow$ **R** be a uniformly continuous function and let $\{x_n\} \subset (a, b)$ be a Cauchy sequence. Prove that $\{G(x_n)\}$ is a Cauchy sequence.

4. $Let \ f(x) = \int_{\pi}^{4\pi} \frac{\sin xt}{t}\, dt$ be defined on **R** and let $x_0 \in$ **R**. Prove, using the definition of continuity, that *f* is continuous at $x = x_0$.

   (*Hint:* You may use, without proof, the lemma for bounding |sin A – sin B|.)

PART IV

Instructions: Select any 4 of the following 5 problems. You may answer all five to obtain extra credit.

1. State and prove the Squeeze Theorem for functions.

2. *Let* $g(x) = \int_0^2 \frac{t^3}{1+x^2t^5+x^4t^9}\,dt.$ Prove that $\lim_{x\to 0} g(x)$ exists and find its value.
Hint: Use the squeeze theorem.

3. Let *f* and *g* be uniformly continuous functions on **R**, and let *a* and *b* be constants. Prove that the linear combination, $h = af + bg$, is also uniformly continuous on **R**.

4. Let f: $\mathbf{R} \to \mathbf{R}$ be a function that satisfies the condition that, for any compact interval *I*, f(I) is a compact interval. Note that we are *not* assuming *f* to be continuous.

(a) Prove that, on any compact interval, *f* achieves a maximum value.

(b) Prove that on any compact interval, *f* satisfies the Intermediate Value Property.

5. Let G: $[0, 1] \to \mathbf{R}$ be continuous. We know from the Boundedness Theorem that $\alpha = \sup G([0, 1])$ exists. Let $\{a_n\}$ be a sequence of points in [0, 1] satisfying

$$G(a_n) > \alpha - 1/n \quad \text{for all integers } n \geq 0.$$

(a) Must it follow that $\{a_n\}$ converge? Explain!

(b) Suppose that $a_n \to p$. Prove that $G(p) = \alpha$.

We are usually convinced more easily by reasons we have found ourselves than by those which occurred to others.

\- Blaise Pascal

This isn't right; this isn't even wrong.

\- Wolfgang Pauli (1900-1958), upon reading a young physicist's paper