# MATH 351 Recent FINAL EXAMINATION

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# April 30th, the 241st birthday of Johann Carl Friedrich Gauß

# PART I *[4 pts each]:* *Answer each of the following 18 questions.*

(a) Give an example of two *convergent* series and  such that  *diverges*.

(b) Give an example of two *divergent* series  and  for which  *converges*.

(c) Let *d* be the Uber metric on *X*, the set of lattice points in the plane,

*viz* X= {(p, q)| p, q **Z**}. If Albertine wishes to travel from A = (1, 1) to B = (3, 5).

1. What is the Uber distance from A to B?
2. List the points that are within the *open* ball, B3((0, 0)), centered at the origin of radius 3.

(d) Give an example of two sequences {an} and {bn} such that {an} *diverges,* {bn} *diverges* and anbn → 2018.

(e) Give an example of a sequence {an} that has countably infinite many cluster points.

(f) Give an example of two functions, f: **R** → **R** andg: **R** → **R** satisfying the conditions that

and

(g) Prove that between any two distinct rational numbers, *p* and *q*, there exists an irrational number.

(h)   Give an example of a *convergent* positive series ∑ an satisfying



(i) Give an example of a sequence that has a strictly increasing subsequence, a strictly decreasing subsequence, and a constant subsequence.

(j) Give an example of a sequence {*In*} of *nested bounded* *non-empty* intervals such that

(k) Let **R** be endowed with the usual Euclidean metric. Give an example of a collection of closed subsets of **R** whose union is not closed.

(l) Given a sequence {cn}, define a new sequence {an} as follows:



(Note that each an is the *moving average* of the last 3 terms of the {cn} sequence.)

Find a *divergent* sequence {cn} for which {an} converges to 11.

(m) Give an example of two bounded sequences {an} and {bn} for which

lim inf anbn ≠ (lim inf an)(lim inf bn)

(n) Give an example of two non-empty bounded sets of real numbers,*A* and *B*, for which

sup AB ≠ (sup A) (sup B). (Here AB is defined to be {ab | a∈A, b∈B}.)

(o) Let f: **R** → **R** be defined by:

For which values of x is *f* continuous?

(p)        Give an example of a *divergent* positive sequence {an} satisfying

= 1

(q)       Give an example of a convergent positive series, ∑ an , such that  diverges.

(r) Give an example of a convergent sequence {an} and a real number ** such that

∀n∈**Z**+ an < ** and lim an ≥ **.

**PART II** *[12 pts each]:**Answer any 9 of the following 11 questions. You may earn extra credit by answering more than 9.*

1. Odette proposes a new metric on the space of all real numbers, X. Albertine protests, and insists that this does not define a true metric.

Here is Odette’s definition:

For any two points P = (a, b) and Q = (c, d), let

D(P, Q) = D((a, b), (c, d)) = (a – c)2 + (b – d)2

1. *[1 pt.]* Prove that D(P, Q) = 0 if P = Q, where
2. *[2 pts]* Prove that D(P, Q) > 0 of
3. *[2 pts]* Prove that D(P, Q) = D(Q, P)
4. *[7 pts]* Show, through a counterexample, that D does not satisfy the triangle inequality.

2.Let *f* : ***R****→* ***R*** be given by f(x) = x2.

(a) *[5 pts]* State as a logical sentence: *f is not uniformly continuous on* ***R****.*

(b) *[7 pts]* Using (a), prove that *f:* ***R****→* ***R*** is not uniformly continuous on ***R***.

(*Hint:* Start by letting = 1.)

3. Prove that the sequence *an = sin n* diverges.

4. Define bn = n1/n. Using the error form principle (or directly), prove that bn→ 1.

(Do *not* use l'Hôpital's rule.)

5. *[4 pts each]* For each of the following numerical series, determine convergence or divergence. You must justify each answer!

(a)

(b)

(c)

6. *[9 pts.]* Using only the *definition of limit,* prove that the following sequence converges:



1. *[3 pts.]* Solve part (b) using the limit theorems for sequences.

7. Prove that the sum of two uniformly continuous functions, each defined on an interval I, is uniformly continuous.

8. Let Find the *three points of discontinuity* of *f*. For each such point, classify the *type* of discontinuity.

9. Let *K* be a cluster point of a sequence {an}. Prove that *K* is the limit of some subsequence of {an}.

10. Let *A* and *B* be bounded non-empty subsets of **R**. Prove that

sup (A+B) ≤ sup A + sup B.

Recall that A+B = {a+b| a∈A, b∈B}.

11. Define g: **R** → **R** as follows: 

1. *[6 pts]* Prove that g is continuous at x = 0.

(b)  *[6 pts]* Is *g* *uniformly continuous* on **R**? Explain.