# MATH 351 Recent FINAL EXAMINATION

#

# April 30th, the 241st birthday of Johann Carl Friedrich Gauß

# PART I *[4 pts each]:* *Answer each of the following 18 questions.*

 (a) Give an example of two *convergent* series and  such that  *diverges*.

 (b) Give an example of two *divergent* series  and  for which  *converges*.

(c) Let *d* be the Uber metric on *X*, the set of lattice points in the plane,

*viz* X= {(p, q)| p, q $\in $ **Z**}. If Albertine wishes to travel from A = (1, 1) to B = (3, 5).

1. What is the Uber distance from A to B?
2. List the points that are within the *open* ball, B3((0, 0)), centered at the origin of radius 3.

(d) Give an example of two sequences {an} and {bn} such that {an} *diverges,* {bn} *diverges* and anbn → 2018.

(e) Give an example of a sequence {an} that has countably infinite many cluster points.

(f) Give an example of two functions, f: **R** → **R** andg: **R** → **R** satisfying the conditions that

$$\lim\_{x\to 3}g\left(x\right)=9 and \lim\_{x\to 9}f\left(x\right)=2018 $$

and $\lim\_{x\to 3} f∘g\left(x\right) exists but does not equal 2018.$

(g) Prove that between any two distinct rational numbers, *p* and *q*, there exists an irrational number.

(h)   Give an example of a *convergent* positive series ∑ an satisfying



(i) Give an example of a sequence that has a strictly increasing subsequence, a strictly decreasing subsequence, and a constant subsequence.

(j) Give an example of a sequence {*In*} of *nested bounded* *non-empty* intervals such that

$$\bigcap\_{n=1}^{\infty }I\_{n}=∅$$

(k) Let **R** be endowed with the usual Euclidean metric. Give an example of a collection of closed subsets of **R** whose union is not closed.

 (l) Given a sequence {cn}, define a new sequence {an} as follows:



(Note that each an is the *moving average* of the last 3 terms of the {cn} sequence.)

Find a *divergent* sequence {cn} for which {an} converges to 11.

 (m) Give an example of two bounded sequences {an} and {bn} for which

lim inf anbn ≠ (lim inf an)(lim inf bn)

 (n) Give an example of two non-empty bounded sets of real numbers,*A* and *B*, for which

sup AB ≠ (sup A) (sup B). (Here AB is defined to be {ab | a∈A, b∈B}.)

(o) Let f: **R** → **R** be defined by:

$$f\left(x\right)=\left\{\begin{array}{c}x^{2}+1 if x\in Q\\17-x^{2} if x \notin Q\end{array}\right.$$

For which values of x is *f* continuous?

(p)        Give an example of a *divergent* positive sequence {an} satisfying

$\lim\_{n\to \infty }\left(a\_{n}\right)^{\frac{1}{n}}$ = 1

(q)       Give an example of a convergent positive series, ∑ an , such that  diverges.

(r) Give an example of a convergent sequence {an} and a real number ** such that

∀n∈**Z**+ an < ** and lim an ≥ **.

**PART II** *[12 pts each]:**Answer any 9 of the following 11 questions. You may earn extra credit by answering more than 9.*

1. Odette proposes a new metric on the space of all real numbers, X. Albertine protests, and insists that this does not define a true metric.

Here is Odette’s definition:

For any two points P = (a, b) and Q = (c, d), let

D(P, Q) = D((a, b), (c, d)) = (a – c)2 + (b – d)2

1. *[1 pt.]* Prove that D(P, Q) = 0 if P = Q, where $P, Q\in X.$
2. *[2 pts]* Prove that D(P, Q) > 0 of $P\ne Q$
3. *[2 pts]* Prove that D(P, Q) = D(Q, P)
4. *[7 pts]* Show, through a counterexample, that D does not satisfy the triangle inequality.

2.Let *f* : ***R****→* ***R*** be given by f(x) = x2.

(a) *[5 pts]* State as a logical sentence: *f is not uniformly continuous on* ***R****.*

 (b) *[7 pts]* Using (a), prove that *f:* ***R****→* ***R*** is not uniformly continuous on ***R***.

(*Hint:* Start by letting $ϵ $= 1.)

3. Prove that the sequence *an = sin n* diverges.

4. Define bn = n1/n. Using the error form principle (or directly), prove that bn→ 1.

(Do *not* use l'Hôpital's rule.)

5. *[4 pts each]* For each of the following numerical series, determine convergence or divergence. You must justify each answer!

(a) $\sum\_{1}^{\infty }\frac{\left(2\right)\left(4\right)\left(6\right)…(2n)}{\left(1\right)\left(3\right)\left(5\right)…(2n-1)}$

(b) $\sum\_{1}^{\infty }\frac{\arctan(n)}{1+n^{2}}$

(c) $\sum\_{1}^{\infty }\frac{1+2^{n}+9^{n}}{5^{n}+8^{n}+n^{2018}}$

6. *[9 pts.]* Using only the *definition of limit,* prove that the following sequence converges:



1. *[3 pts.]* Solve part (b) using the limit theorems for sequences.

7. Prove that the sum of two uniformly continuous functions, each defined on an interval I, is uniformly continuous.

8. Let $f\left(x\right)=\frac{x^{4}-16}{x^{3}+x^{2}-6x} .$ Find the *three points of discontinuity* of *f*. For each such point, classify the *type* of discontinuity.

9. Let *K* be a cluster point of a sequence {an}. Prove that *K* is the limit of some subsequence of {an}.

10. Let *A* and *B* be bounded non-empty subsets of **R**. Prove that

sup (A+B) ≤ sup A + sup B.

Recall that A+B = {a+b| a∈A, b∈B}.

11. Define g: **R** → **R** as follows: 

1. *[6 pts]* Prove that g is continuous at x = 0.

(b)  *[6 pts]* Is *g* *uniformly continuous* on **R**? Explain.