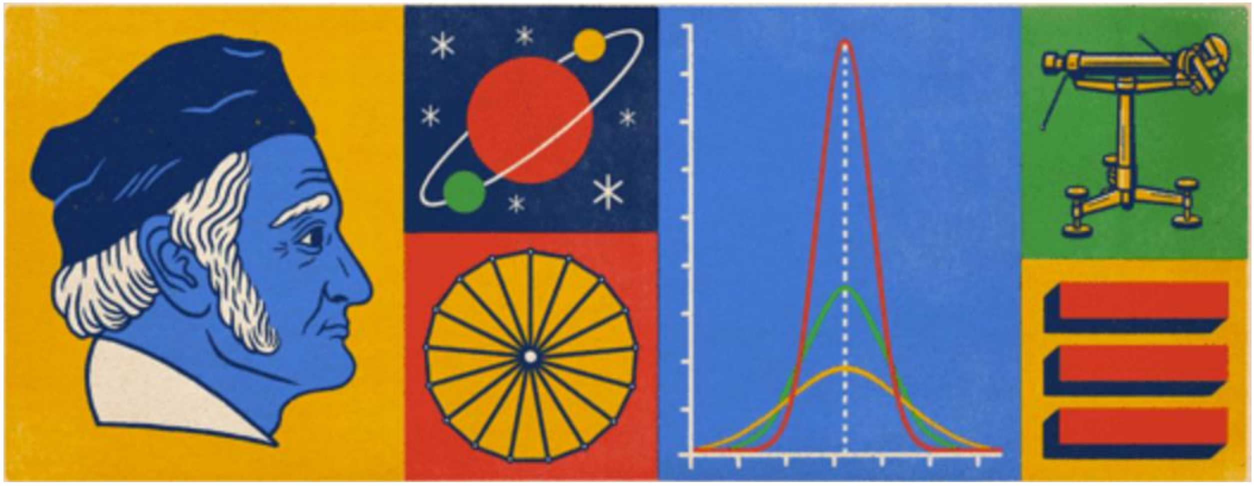
MATH 351 RECENT FINAL EXAMINATION



April 30th, the 241st birthday of Johann Carl Friedrich Gauß

PART I *[4 pts each]: Answer each of the following 18 questions.*

(a) Give an example of two *convergent* series $\sum_n a_n$ and $\sum_n b_n$ such that $\sum_n a_n b_n$ *diverges*.

(b) Give an example of two *divergent* series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ for which $\sum_{n=1}^{\infty} a_n b_n$ *converges*.

(c) Let d be the Uber metric on X, the set of lattice points in the plane,

viz X= {(p, q)| p, q ∈ **Z**}. If Albertine wishes to travel from A = (1, 1) to B = (3, 5).

(i) What is the Uber distance from A to B?

(ii) List the points that are within the *open* ball, $B_3((0, 0))$, centered at the origin of radius 3.

(d) Give an example of two sequences $\{a_n\}$ and $\{b_n\}$ such that $\{a_n\}$ *diverges*, $\{b_n\}$ *diverges* and $a_n b_n \to 2018$.

(e) Give an example of a sequence $\{a_n\}$ that has countably infinite many cluster points.

(f) Give an example of two functions, f: **R** → **R** and g: **R** → **R** satisfying the conditions that

$$\lim_{x\to 3} g(x) = 9 \textit{ and } \lim_{x\to 9} f(x) = 2018$$

and $\lim_{x\to 3} f\circ g(x)$ *exists but does not equal* 2018.

(g) Prove that between any two distinct rational numbers, *p* and *q*, there exists an irrational number.

(h) Give an example of a *convergent* positive series $\sum a_n$ satisfying

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1$$

(i) Give an example of a sequence that has a strictly increasing subsequence, a strictly decreasing subsequence, and a constant subsequence.

(j) Give an example of a sequence $\{I_n\}$ of *nested bounded non-empty* intervals such that

$$\bigcap_{n=1}^{\infty} I_n = \emptyset$$

(k) Let **R** be endowed with the usual Euclidean metric. Give an example of a collection of closed subsets of **R** whose union is not closed.

(l) Given a sequence $\{c_n\}$, define a new sequence $\{a_n\}$ as follows:

$$a_n = \frac{c_{n-2} + c_{n-1} + c_n}{3} \quad for\ n \geq 3$$

(Note that each a_n is the *moving average* of the last 3 terms of the $\{c_n\}$ sequence.)

Find a *divergent* sequence $\{c_n\}$ for which $\{a_n\}$ converges to 11.

(m) Give an example of two bounded sequences $\{a_n\}$ and $\{b_n\}$ for which

$$\liminf a_n b_n \neq (\liminf a_n)(\liminf b_n)$$

(n) Give an example of two non-empty bounded sets of real numbers, *A* and *B*, for which $\sup AB \neq (\sup A)(\sup B)$. (Here AB is defined to be $\{ab \mid a \in A, b \in B\}$.)

(o) Let f: **R** → **R** be defined by:

$$f(x) = \begin{cases} x^2 + 1 & if\ x \in Q \\ 17 - x^2 & if\ x \notin Q \end{cases}$$

For which values of x is *f* continuous?

(p) Give an example of a *divergent* positive sequence $\{a_n\}$ satisfying

$$\lim_{n\to\infty} (a_n)^{\frac{1}{n}} = 1$$

(q) Give an example of a convergent positive series, $\sum a_n$, such that $\sum \sqrt{\frac{a_n}{n}}$ diverges.

(r) Give an example of a convergent sequence $\{a_n\}$ and a real number β such that

$\forall n \in \mathbf{Z}^+ \quad a_n < \beta$ and $\lim a_n \geq \beta$.

PART II *[12 pts each]: Answer any 9 of the following 11 questions. You may earn extra credit by answering more than 9.*

1. Odette proposes a new metric on the space of all real numbers, X. Albertine protests, and insists that this does not define a true metric.
   Here is Odette's definition:

$$\text{For any two points } P = (a, b) \text{ and } Q = (c, d), \text{ let}$$
$$D(P, Q) = D((a, b), (c, d)) = (a - c)^2 + (b - d)^2$$

(a) *[1 pt.]* Prove that $D(P, Q) = 0$ if $P = Q$, where $P, Q \in X$.

(b) *[2 pts]* Prove that $D(P, Q) > 0$ of $P \neq Q$

(c) *[2 pts]* Prove that $D(P, Q) = D(Q, P)$

(d) *[7 pts]* Show, through a counterexample, that D does not satisfy the triangle inequality.

2. Let $f : \boldsymbol{R} \rightarrow \boldsymbol{R}$ be given by $f(x) = x^2$.

(a) *[5 pts]* State as a logical sentence: *f is not uniformly continuous on* $\boldsymbol{R}$.

(b) *[7 pts]* Using (a), prove that $f: \boldsymbol{R} \rightarrow \boldsymbol{R}$ is not uniformly continuous on $\boldsymbol{R}$.
(*Hint:* Start by letting $\epsilon = 1$.)

3. Prove that the sequence $a_n = \sin n$ diverges.

4. Define $b_n = n^{1/n}$. Using the error form principle (or directly), prove that $b_n \to 1$. (Do *not* use l'Hôpital's rule.)

5. *[4 pts each]* For each of the following numerical series, determine convergence or divergence. You must justify each answer!

(a) $\sum_1^\infty \frac{(2)(4)(6)\ldots(2n)}{(1)(3)(5)\ldots(2n-1)}$

(b) $\sum_1^\infty \frac{\arctan n}{1+n^2}$

(c) $\sum_1^\infty \frac{1+2^n+9^n}{5^n+8^n+n^{2018}}$

6. *[9 pts.]* Using only the *definition of limit,* prove that the following sequence converges:

$$b_n = \frac{n^4 + 5n^2 + 8}{n^4 + n + 2}$$

(b) *[3 pts.]* Solve part (b) using the limit theorems for sequences.

7. Prove that the sum of two uniformly continuous functions, each defined on an interval I, is uniformly continuous.

8. Let $f(x) = \frac{x^4-16}{x^3+x^2-6x}$. Find the *three points of discontinuity* of *f*. For each such point, classify the *type* of discontinuity.

9. Let *K* be a cluster point of a sequence $\{a_n\}$. Prove that *K* is the limit of some subsequence of $\{a_n\}$.

10. Let *A* and *B* be bounded non-empty subsets of **R**. Prove that

$$\sup(A+B) \le \sup A + \sup B.$$

Recall that $A+B = \{a+b \mid a \in A,\ b \in B\}$.

11. Define g: **R** $\to$ **R** as follows: $$g(x) = \begin{cases} x^2 \sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

(a) *[6 pts]* Prove that g is continuous at x = 0.

(b) *[6 pts]* Is *g uniformly continuous* on **R**? Explain.