# MATH 351 additional Practice FINAL EXAMINATION questions (from past exams) (REVISED)

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# PART I

 (a) Give an example of two convergent series an and bn such that anbn diverges.

(b) Give an example of a continuous function, g: (0, ∞) → **R**, such that *g* is uniformly continuous on (1, ∞), but not uniformly continuous on (0, 1].

(c) Find a function g: **R** → **R** such that g is continuous everywhere except for the set {1/n : n∈**N**}.

(d) Give an example of a sequence that has a strictly increasing subsequence, a strictly decreasing subsequence and a constant subsequence.

(e) Give an example of a sequence {tn} such that tn → 0 and sin(1/tn) → ½.

(f) Give an example of a sequence {In} of nested closed intervals such that



(g) Find a sequence {bn} such that, for all *p* ∈ **N**, there exists a subsequence of {bn} that converges to *p*.

(h) Given a sequence {cn}, define a new sequence {an} as follows:



(Note that each an is the average of the first *n* terms of the {cn} sequence.)

Find a divergent sequence {cn} for which {an} converges.

 (i) Give an example of a function f: **R** → **R** which is *not* continuous everywhere, yet which satisfies the property that: if *I* is any compact interval then f(I) is a compact interval.

(j) Give an example of two series an and bn for which (an + bn) converges and yet the series a1 + b1 + a2 + b2 + a3 + b3 + … diverges.

(k) Give an example of two bounded sequences {an} and {bn} for which

lim sup anbn ≠ (lim sup an)(lim sup bn)

 (l) Give an example of two non-empty sets of real numbers,A and B, for which sup AB ≠ (sup A)(sup B). *(Here AB is defined to be {ab| a∈A, b∈B}.)*

(m) Let f: **R** → **R** be defined by:



Where is *f* continuous?

(o) Give an example of a function that is bounded and continuous but not uniformly continuous.

(p) Give an example of two divergent sequences {an} and {bn} such that the sequence {anbn} is a Cauchy sequence.

(q) Give an example of a function g: **R** → **R** that is continuous only at x = 0.

1. Give an example of two continuous functions, f(x) and g(x), each defined on (-∞, 5), satisfying:

f(x) < g(x) for all x < 5

and



 (s) Give an example of a continuous function f: (0,1) → **R** and a Cauchy sequence {an} ⊂ (0,1) such that {f(an)} does not converge.

**PART II**

1. (a) State the Comparison Test for numerical series.

(b) Determine convergence or divergence of the following series. Prove your result using the Comparison Test.



2. (a) Let A be a non-empty subset of **R**. Define *uniform continuity* for a function F: A→ **R**.

(b) State what it means for a function F: A→ **R** to *fail to be uniformly continuous* on A by negating your definition above**.**

3. (a) State the Nested Interval Theorem.

 (b) Using the Nested Interval Theorem, give Georg Cantor’s original proof that **R** is uncountable.

4. Prove, using *only the definition of continuity*, that  is continuous

at x = 2.

5. Prove that

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6. Let {an} and {bn} be sequences satisfying an+bn → A and an-bn → B. Prove that {anbn} converges, and that its limit is .

7. Prove that ln n! ~ n ln n.

8. State and prove the *nth term test* for divergence of a series.

 and let .

9. Find each of the following:

1. lim inf an
2. lim inf bn
3. lim sup an
4. lim sup bn

10. f: I → **R** is said to be a *Lipschitz function* if there exists L > 0 such that for all *x*, *t* in I, |f(x) – f(t)| < L|x-t|.

1. Prove that if *f* is a Lipschitz function then *f* is uniformly continuous
2. Give an example of a uniformly continuous function that is not Lipschitz.

11. The *Dirichlet function* f: (0,1) → **R** is defined by:

 

Prove that, for all a∈(0, 1),



(*Hint:* Show that, for all q ∈ **N**, there exist only finitely many x ∈ (0, 1) such that f(x) ≥ 1/q.)

12. Let *n\** be the smallest positive integer *n* for which the inequality (1 + x)n > 1 + nx + nx2 is true for all x > 0. Compute *n\** and prove that the inequality is true for all integers n ≥ *n\**.

13. *Thomae’s function* (aka the *Popcorn Function*) f: (0, 1) → **R** is defined by:



Prove that, for all a∈(0, 1),



(*Hint:* Show that, for all q ∈ N, there exist only finitely many x ∈ (0, 1) such that f(x) ≥ 1/q.)

**PART III**

1. Using mathematical induction prove the following:

1(1!) + 2(2!) + 3(3!) + … + n(n!) = (n + 1)! – 1 for all n∈ **N**.

2. Let *E* be a non-empty subset of **R** that is bounded above. Define

*U* = { ∈ **R** :  is an upper bound of *E*}.

Prove that sup *E* = inf *U*.

3. Let *f* be a continuous function defined on[0, 13]. Prove that *f* is bounded.

4. Let *f* and *g* be uniformly continuous functions defined on **R**. Prove that the composition function, , is also uniformly continuous on **R**.

5. Let an > 0 for all n ∈ **N**. If converges, prove that  converges.

*Hint:* Expand 

6. Let an = n1/n. Prove that an → 1.

7. Let {pn} be a Cauchy sequence that has a convergent subsequence. Prove that {pn} converges.