MATH 351 ADDITIONAL PRACTICE FINAL EXAMINATION QUESTIONS (FROM PAST EXAMS) (REVISED)

PART I

(a) Give an example of two convergent series $\Sigma\, a_n$ and $\Sigma\, b_n$ such that $\Sigma\, a_n b_n$ diverges.

(b) Give an example of a continuous function, g: $(0, \infty) \to \mathbf{R}$, such that g is uniformly continuous on $(1, \infty)$, but not uniformly continuous on $(0, 1]$.

(c) Find a function g: $\mathbf{R} \to \mathbf{R}$ such that g is continuous everywhere except for the set $\{1/n : n \in \mathbf{N}\}$.

(d) Give an example of a sequence that has a strictly increasing subsequence, a strictly decreasing subsequence and a constant subsequence.

(e) Give an example of a sequence $\{t_n\}$ such that $t_n \to 0$ and $\sin(1/t_n) \to ½$.

(f) Give an example of a sequence $\{I_n\}$ of nested closed intervals such that

$$\bigcap_{n=1}^{\infty} I_n = \phi$$

(g) Find a sequence $\{b_n\}$ such that, for all $p \in \mathbf{N}$, there exists a subsequence of $\{b_n\}$ that converges to p.

(h) Given a sequence $\{c_n\}$, define a new sequence $\{a_n\}$ as follows:

$$a_n = \frac{c_1 + c_2 + c_3 + \ldots + c_n}{n}$$

(Note that each a_n is the average of the first n terms of the $\{c_n\}$ sequence.)
Find a divergent sequence $\{c_n\}$ for which $\{a_n\}$ converges.

(i) Give an example of a function f: $\mathbf{R} \to \mathbf{R}$ which is *not* continuous everywhere, yet which satisfies the property that: if I is any compact interval then f(I) is a compact interval.

(j) Give an example of two series $\Sigma\, a_n$ and $\Sigma\, b_n$ for which $\Sigma\, (a_n + b_n)$ converges and yet the series $a_1 + b_1 + a_2 + b_2 + a_3 + b_3 + \ldots$ diverges.

(k) Give an example of two bounded sequences $\{a_n\}$ and $\{b_n\}$ for which

$$\limsup a_n b_n \neq (\limsup a_n)(\limsup b_n)$$

(m) Give an example of two non-empty sets of real numbers, A and B, for which

sup AB $\neq$ (sup A)(sup B). *(Here AB is defined to be $\{ab \mid a \in A, b \in B\}$.)*

(n) Let f: **R** $\rightarrow$ **R** be defined by:

$$f(x) = \begin{cases} x^2 & \text{if } x \in Q \\ x+2 & \text{if } x \notin Q \end{cases}$$

Where is *f* continuous?

(o) Give an example of a function that is bounded and continuous but not uniformly continuous.

(p) Give an example of two divergent sequences $\{a_n\}$ and $\{b_n\}$ such that the sequence $\{a_n b_n\}$ is a Cauchy sequence.

(q) Give an example of a function g: **R** $\rightarrow$ **R** that is continuous only at x = 0.

(r) Give an example of two continuous functions, f(x) and g(x), each defined on $(-\infty, 5)$, satisfying:

$$f(x) < g(x) \text{ for all } x < 5$$

and

$$\lim_{x\to 5} f(x) = \lim_{x\to 5} g(x)$$

(s) Give an example of a continuous function f: (0,1) $\rightarrow$ **R** and a Cauchy sequence

$\{a_n\} \subset (0,1)$ such that $\{f(a_n)\}$ does not converge.

PART II

1. (a) State the Comparison Test for numerical series.
   (b) Determine convergence or divergence of the following series. Prove your result using the Comparison Test.

$$\sum_{n=1}^{\infty} \frac{(n^2+1789)(n+2008)}{(n+1776)^2(n+1492)^2}$$

2. (a) Let A be a non-empty subset of **R**. Define *uniform continuity* for a function F: A$\rightarrow$ **R**.
   (b) State what it means for a function F: A$\rightarrow$ **R** to *fail to be uniformly continuous* on A by negating your definition above**.**
3. (a) State the Nested Interval Theorem.
   (b) Using the Nested Interval Theorem, give Georg Cantor's original proof that **R** is uncountable.

4. Prove, using *only the definition of continuity*, that $f(x) = \dfrac{x^2+4}{x+2}$ is continuous at x = 2.

5. Prove that

$$\lim_{n\to\infty} \int_0^{\frac{\pi}{2}} \cos^n x \, dx = 0 .$$

6. Let $\{a_n\}$ and $\{b_n\}$ be sequences satisfying $a_n+b_n \to A$ and $a_n-b_n \to B$. Prove that $\{a_nb_n\}$ converges, and that its limit is $\frac{1}{4}(A^2 - B^2)$.

7. Prove that ln n! ~ n ln n.

8. State and prove the *n^{th} term test* for divergence of a series.

$a_n = (-1)^n n \sin(1/n) + \dfrac{n^3+1}{(n+1)^3}$ and let $b_n = \dfrac{(-1)^n n^2+1}{(2n+1)^2}$.

9. Find each of the following:
   (A) lim inf a_n
   (B) lim inf b_n
   (C) lim sup a_n
   (D) lim sup b_n

10. f: I $\to$ **R** is said to be a *Lipschitz function* if there exists L > 0 such that for all *x*, *t* in I, |f(x) – f(t)| < L|x-t|.

   (A) Prove that if *f* is a Lipschitz function then *f* is uniformly continuous
   (B) Give an example of a uniformly continuous function that is not Lipschitz.

11. The *Dirichlet function* f: (0,1) $\to$ **R** is defined by:

$$f(x) = \begin{cases} \dfrac{1}{q} & \text{if } x = \dfrac{p}{q} \text{ is in lowest terms with } p, q \in N \\ 0 & \text{if } x \text{ is irrational} \end{cases}$$

Prove that, for all a$\in$(0, 1),

$$\lim_{x\to a} f(x) = 0.$$

(*Hint:* Show that, for all q $\in$ **N**, there exist only finitely many x $\in$ (0, 1) such that f(x) $\geq$ 1/q.)

12. Let n^* be the smallest positive integer n for which the inequality $(1 + x)^n > 1 + nx + nx^2$ is true for all $x > 0$. Compute n^* and prove that the inequality is true for all integers $n \geq n^*$.

13. *Thomae's function* (aka the *Popcorn Function*) f: (0, 1) → **R** is defined by:

$$f(x) = \begin{cases} \frac{1}{q} & if \;\; x = \frac{p}{q} \;\; is \; in \; lowest \; terms \; with \;\; p, q \in N \\ 0 & if \;\; x \;\; is \;\; irrational \end{cases}$$

Prove that, for all $a \in (0, 1)$,

$$\lim_{x \to a} f(x) = 0.$$

(*Hint:* Show that, for all $q \in N$, there exist only finitely many $x \in (0, 1)$ such that $f(x) \geq 1/q$.)

PART III

1. Using mathematical induction prove the following:

$$1(1!) + 2(2!) + 3(3!) + \ldots + n(n!) = (n + 1)! - 1 \;\; \text{for all } n \in \mathbf{N}.$$

2. Let E be a non-empty subset of **R** that is bounded above. Define

$$U = \{\beta \in \mathbf{R} : \beta \text{ is an upper bound of } E\}.$$

Prove that $\sup E = \inf U$.

3. Let f be a continuous function defined on [0, 13]. Prove that f is bounded.

4. Let f and g be uniformly continuous functions defined on **R**. Prove that the composition function, $g \circ f$, is also uniformly continuous on **R**.

5. Let $a_n \geq 0$ for all $n \in \mathbf{N}$. If $\sum_{n=1}^{\infty} a_n$ converges, prove that $\sum_{n=1}^{\infty} \frac{\sqrt{a_n}}{n}$ converges.

Hint: Expand $\left(\sqrt{a_n} - \frac{1}{n}\right)^2$

6. Let $a_n = n^{1/n}$. Prove that $a_n \to 1$.

7. Let $\{p_n\}$ be a Cauchy sequence that has a convergent subsequence. Prove that $\{p_n\}$ converges.