

MATH 351: 27 AUGUST 2018

$$\sum_{n=1}^{\infty} n \rightarrow -\frac{1}{12}$$

What is that?

The **Ramanujan sum** of $1 + 2 + 3 + 4 + \dots$ is $-1/12$. Here is what Ramanujan wrote in his second letter to G. H. Hardy, dated 27 February 1913:

Dear Sir,

I am very much gratified on perusing your letter of the 8th February 1913. I was expecting a reply from you similar to the one which a Mathematics Professor at London wrote asking me to study carefully Bromwich's *Infinite Series* and not fall into the pitfalls of divergent series. ... I told him that the sum of an infinite number of terms of the series: $1 + 2 + 3 + 4 + \dots = -1/12$ under my theory. If I tell you this, you will at once point out to me the lunatic asylum as my goal. I dilate on this simply to convince you that you will not be able to follow my methods of proof if I indicate the lines on which I proceed in a single letter. ...

Cf.: [Video: Infinity or just -1/12 ?](#)

In the End, It All Adds Up to $-1/12$

Dennis Overbye, **New York Times**

OUT THERE FEB. 3, 2014

This is what happens when you mess with infinity.

You might think that if you simply started adding the natural numbers, 1 plus 2 plus 3 and so on all the way to infinity, you would get a pretty big number. At least I always did.

So it came as a shock to a lot of people when, in a recent [video](#), a pair of physicists purported to prove that this infinite series actually adds up to ... minus $1/12$.

To date some 1.5 million people have viewed this calculation, which plays a key role in modern physics and quantum theory; the answer, as absurd as it sounds, has been verified to many decimal places in lab experiments. After watching the video myself, I checked to make sure I still had my wallet and my watch.

Even the makers of the video, [Brady Haran](#), a journalist, and Ed Copeland and [Antonio Padilla](#), physicists at the University of Nottingham in England, admit there is a certain amount of "hocus-pocus," or what some mathematicians have called dirty tricks, in their presentation. Which has led to some online grumbling.

ASTOUNDING: $1 + 2 + 3 + 4 + 5 + \dots = -1/12$

But there is broad agreement that a more rigorous approach to the problem gives the same result, as shown by a formula in [Joseph Polchinski's](#) two-volume textbook "[String Theory](#)."

So what's going on with infinity?

"This calculation is one of the best-kept secrets in math," said [Edward Frenkel](#), a mathematics professor at the University of California, Berkeley, and author of "[Love and Math: The Heart of Hidden Reality](#)," (Basic Books, 2013), who was in town recently promoting his book and acting as an ambassador for better math education. "No one on the outside knows about it."

The great 18th-century mathematician [Leonhard Euler](#), who was born in Switzerland but did most of his work in Berlin and St. Petersburg, Russia, was the first one down this road. Euler wanted to know if you could find an answer to endless sums of numbers like 1 plus 1/2 plus 1/3 plus 1/4 on up to infinity, or the squares of those fractions.

These are all different versions of what has become known as the [Riemann zeta function](#), after [Bernhard Riemann](#), who came along about a century after Euler. The zeta function is one of the more mysterious and celebrated subjects in mathematics, important in the theory of prime numbers, among other things. It was one of the plot threads, for example, in Thomas Pynchon's 2006 novel, "[Against the Day](#)."

In 1749, Euler used a bag of mathematical tricks to solve the problem of adding the natural numbers from 1 to infinity, a so-called divergent series because the terms keep growing without limit as you go along. Clearly, if you stop adding anywhere along the way — at a quintillion (1 with 18 zeros after it), say, or a googolplex (10^{100} zeros) — the sum will be enormous. The problem with infinity is that you can't stop. You never get there. It's more of a journey than a destination. As Dr. Padilla says to Mr. Haran at the end of their video, "You have to face infinity, Brady."

The method in the video is essentially the same as Euler's. It involves nothing more complicated than addition and subtraction (although the things being added and subtracted were more infinite series) and a small piece of algebra that my sixth-grade daughter would breeze through.

You are not alone in wondering how this can make sense. The Norwegian mathematician [Niels Henrik Abel](#), whose notion of an Abel sum plays a role here, once wrote, "The divergent series are the invention of the devil, and it is a shame to base on them any demonstration whatsoever."

In modern terms, Dr. Frenkel explained, the gist of the calculations can be interpreted as saying that the infinite sum has three separate parts: one of which blows up when you go to infinity, one of which goes to zero, and minus 1/12. The infinite term, he said, just gets thrown away.

And it works. A hundred years later, Riemann used a more advanced and rigorous method, involving imaginary as well as real numbers, to calculate the zeta function and got the same answer: minus 1/12.

"So Euler guessed it right," Dr. Frenkel said.

Those of us who are not mathematicians probably wouldn't care so much about infinity except that it crops up again and again in calculations of things, like the energy of the electron, that we know are finite, or in string theory, which physicists would like to hope is finite.

In this case, our current understanding of the very solidity of reality depends on coming up with a consistent way to assign values to infinite sums.

In the process known as [regularization](#), which is a part of many calculations in quantum theory, physicists do something similar to what Euler did, arriving at a real number that corresponds to the quantity they want to know and an infinite term, which they throw away. The process works so well that theoretical predictions in quantum electrodynamics, the fancy version of the familiar force of electromagnetism, agree with experiments to a precision of one part in a trillion.

Which is remarkable given that infinite quantities have been thrown away, or "swept under the rug," in the words of the California Institute of Technology physicist [Richard Feynman](#), who helped invent a lot of this stuff but thought it was more than faintly scandalous.

Likewise, it is no surprise that the factor 1/12 shows up a lot in string theory equations, Dr. Frenkel said. Why it all works is still a mystery.

"Quantum physics needs its own Riemann to come and give a rigorous explanation of these mysteries," Dr. Frenkel said.

To him and others, this is just another example of what the eminent physicist [Eugene Wigner](#) called the "unreasonable effectiveness of mathematics." Why should such woolly and abstract concepts as

zeta functions or imaginary numbers, the products of a chess game in our minds, have such relevance in describing the world?

Riemann's explorations of the geometry of curved spaces in 1854 laid the foundation for Einstein's theory of gravity, general relativity, half a century later.

There were mathematicians and philosophers who were ready to jump out the window later in the 1800s when [Georg Cantor](#), a Russian-born mathematician, set out to classify the kinds of infinity. In a speech in 1908, the French mathematician [Henri Poincaré](#) compared "Cantorism," as he called it, to a disease.

Mathematicians today agree that there is an infinite number of natural numbers (1, 2, 3 and so on) on the bottom rung of infinity. Above that, however, is another rung of so-called real numbers, which is bigger in the sense that there is an uncountable number of them for every natural number. And so it goes.

Cosmologists do not know if the universe is physically infinite in either space or time or what it means if it is or isn't. Or if these are even sensible questions. They don't know whether someday they will find that higher orders of infinity are unreasonably effective in understanding existence, whatever that is. Here is where we sprain our imaginations and perhaps check to see that we still have our wallets.

