MATH 351 SOLUTIONS: TEST 1 1 OCTOBER 2018

PART A: Definitions & Statements of Theorems

[10 points each] Be precise and careful.

1. Carefully state the **Quotient Theorem** for sequences.

Let $\{a_n\}$ *and* $\{b_n\}$ be sequences and assume that $a_n \to L$ *and* $b_n \to M$.

Then $\frac{a_n}{b_n} \to \frac{L}{M}$ provided that $L \neq 0$ *and* $b_n \neq 0$ *for all* n.

2. State the **general triangle inequality**. Aka (the extended triangle inequality).

Let $a_1, a_{2,} \ldots a_n \in R$. *Then* $|a_1 + a_2 + \cdots + a_n| \leq |a_1| + |a_2| + \cdots + |a_n|$

3. Let $\{a_n\}$ *be a sequence*, L *be real*, *and* $\varepsilon > 0$.

Define $\boldsymbol{a_n \underset{\varepsilon}{\approx} L \; for \; n \gg 1}$.

$\boldsymbol{|a_n - L| < \varepsilon \; for \; n \gg 1}$

4. State the ***Subsequence Theorem***.

Let $\{a_n\}$ *be a convergent sequence, with limit* L. *Then every subsewuence of* $\{a_n\}$ *converges to* L.

5. State the ***Nested Intervals Theorem***.

Let $I_n = [a_n, b_n]$ *be a sequence of nested interals, that is, for all* $n \geq 1$ $[a_{n+1}, b_{n+1}] \subseteq [a_n, b_n]$, *and assume that* $b_n - a_n \to 0$.

Then there exists a unique number L *such that* $L \in \bigcap_{n=1}^{\infty} I_n$. *Moverover,* $a_n \to L$ *and* $b_n \to L$.

6. Define $\boldsymbol{\lim_{n\to\infty} a_n} = \infty$.

$\lim_{n\to\infty} a_n = \infty$ *means that* $\forall M$ $\boldsymbol{a_n > M \; for \; n \gg 1}$.

7. State the ***Sequence Location Theorem***.

Let $\{a_n\}$ *be a convergent sequence.Then*

$\lim a_n < M \Rightarrow a_n < M \; for \; n \gg 1$

and $\lim a_n > M \Rightarrow a_n > M \; for \; n \gg 1$.

PART B: *True or False* *[6 points each]*

Determine if each of the following statements is *True* or *False*. If False, provide a *precise counter-example*; if True, give a *very brief* justification.

i If $\{a_n\}$ and $\{b_n\}$ are sequences that are each bounded below, then so is the sequence $\{c_n\}$ defined by $c_n = a_n + b_n$

TRUE: *Since $\{a_n\}$ and $\{b_n\}$ are sequences that are each bounded below, there exist M and K such that*

$$a_n \geq M \text{ and } b_n \geq K \quad \forall n \geq 1$$

Hence

$$a_n + b_n \geq M + K \quad \forall n \geq 1$$

ii Let $\{a_n\}$ and $\{b_n\}$ be sequences such that $\{a_n + b_n\}$ converges. Then the sequences $\{a_n\}$ and $\{b_n\}$ each converge.

FALSE:

Let $a_n = n$ and $b_n = -n$ for all n. Clearly $\{a_n + b_n\}$ converges yet $\{a_n\}$ fails to converge..

iii Let $\{a_n\}$ and $\{b_n\}$ be sequences, such that the sequence $\{a_n + b_n\}$ diverges. Then either $\{a_n\}$ or $\{b_n\}$ (or possibly both) diverges.

TRUE:

The contrapositive is: If both $\{a_n\}$ and $\{b_n\}$ converge, then $\{a_n + b_n\}$ converges. This follows from the limit law for sums.

iv Let $\{a_n\}$ and $\{b_n\}$ be sequences for which each of $\{2a_n + 3b_n\}$ and $\{4a_n - 5b_n\}$ converges. Then the sequences $\{a_n\}$ and $\{b_n\}$ each converge.

TRUE: *Note that* $a_n = \frac{5(2a_n+3b_n)+3(4a_n-5b_n)}{22} = \frac{1}{22}\ (5(2a_n + 3b_n) + 3(4a_n - 5b_n))$

So $\lim a_n = 5\lim(2a_n + 3b_n) + 3\lim(4a_n - 5b_n)$.

We can use a similar argument to show that $\{b_n\}$ converges.

v If $\{a_n\}$ converges to 0, then $\{|a_n|\}$ converges to 0.

TRUE: *Let* $\varepsilon > 0$. *Then* $|a_n - 0| < \varepsilon$ *for* $n \gg 1$ *since* $a_n \to 0$. *Now* $|\,|a_n| - 0| = |a_n - 0| < \varepsilon$.

Hence $|a_n| \to 0$.

vi Suppose that $\{a_n\}$ and $\{b_n\}$ are sequences satisfying $0 < a_n < b_n$ for all $n \in \mathbf{Z}^+$. Then, if $\{a_n\}$ diverges, it follows that $\{b_n\}$ diverges.

FALSE: *Let $\{a_n\}$ be the sequence, 1, 2, 1, 2, 1, 2, …*

Let $\{b_n\}$ be the sequence 3, 3, 3, 3, …

Then $0 < a_n < b_n$ for all n , $\{a_n\}$ diverges, yet $\{b_n\}$ converges.

vii Consider a sequence $\{a_n\}$ for which the sequence $\left\{\frac{a_n}{\sqrt{n}}\right\}$ converges. Then $\{a_n\}$ converges.

FALSE: Let $a_n = \sqrt{n}$. Then $\frac{a_n}{\sqrt{n}} = 1 \text{ for all } n$.

Hence $\left\{\frac{a_n}{\sqrt{n}}\right\}$ *converges, yet* $\{a_n\}$ *diverges.*

viii Let $\{a_n\}$ be a convergent sequence satisfying the condition: $a_n < M$ for n>>1. Then $\lim_{n\to\infty} a_n < M$.

FALSE: *Let M = 1 and* $a_n = 1 - \frac{1}{n}$ *for all n. Now* $a_n < M$ *and yet* $\lim_{n\to\infty} a_n = 1 + M$.

ix Let $\{a_n\}$ and $\{b_n\}$ be sequences such that $\{a_n + b_n\}$ converges to 0. Then $\{a_n\}$ and $\{b_n\}$ are bounded.

FALSE: *Let* $a_n = n$ *and* $b_n = -n$ *for all n. Clearly nether* $\{a_n\}$ *nor* $\{b_n\}$ *is bounded.*

x *Let* $\{a_n\}$ be a sequence of positive real numbers for which $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 0$. Then $\{a_n\}$ converges.

TRUE: Since $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 0, \frac{a_{n+1}}{a_n} < \frac{1}{2} \text{ for } n \gg 1$. *In other words,* $a_{n+1} < \frac{1}{2} a_n \text{ for } n \gg 1$.

From this, we can deduce that $a_n \to 0$.

PART C: PROOFS *[16 points each]*

Instructions: *Select any 3 of the following 5 problems. You may answer a fourth question to earn extra credit. Do not answer more than 4.*

1. Define the sequence $\{a_n\}$ by $a_n = \frac{n^2-3n-1}{(n+1)^2}$. Guess the limit, L, of $\{a_n\}$ and prove, *using only the definition of limit*, that $\{a_n\}$ converges to L.

Proof: *Since* $\frac{n^2-3n-1}{(n+1)^2} \approx \frac{n^2}{n^2} = 1 \text{ for } n \gg 1, \text{we guess that the limit}, L, \text{is } 1.$

Next, let $\varepsilon > 0$. *Choose* $N^* = \frac{7}{\varepsilon}$

Then $n > N^* \Rightarrow \left|\frac{n^2-3n-1}{(n+1)^2} - 1\right| = \left|\frac{n^2-3n-1-(n+1)^2}{(n+1)^2}\right| = \left|\frac{-5n-2}{(n+1)^2}\right| =$

$\frac{5n+2}{(n+1)^2} < \frac{5n+2n}{n^2} = \frac{7}{n} < \frac{7}{N^*} < \varepsilon$ *since* $N^* \geq \frac{7}{\varepsilon}$.

2. Prove that if the sequence $\{a_n\}$ *converges, then its limit is unique.*

Proof: *Suppose, contrary to fact, that* $b_n \to L_1$ *and* $b_n \to L_2$ *where* $L_1 \neq L_2$

Now choose $\varepsilon = \frac{1}{2}\ |L_1 - L_2|$

So for n>>1, $|L_1 - a_n| < \varepsilon$ *and* $|L_2 - a_n| < \varepsilon$.

Using the triangle inequality, $|L_1 - L_2| < \left|(L_1 - a_n) + (a_n - L_2)\right| < |L_1 - a_n| + |L_2 - a_n| <$

$2\varepsilon < 2\left(\frac{1}{2}\right)\ |L_1 - L_2| = |L_1 - L_2|$

But this means that $|L_1 - L_2| > 2\varepsilon = 2\left(\frac{1}{2}\ |L_1 - L_2|\right) = |L_1 - L_2|$

which contradicts our initial assumption. Hence $L_1 = L_2$.

3. Prove that if a > 1, *then* $a^n \to \infty$.

Proof: *Since* $a > 1$, $a = 1 + h$ *where* $h > 0$.
Using Bernoulli's inequality, we have $a^n > 1 + nh > nh$.
So, given any $M > 0$, *choose* $N^* = \frac{M}{h}$.
Now, when $n > N^* = \frac{M}{h}$, *it follows that* $a^n > a^{N^*} > N^* h = M$.
Hence, by definition, $a^n \to \infty$.

4. State and prove the *Limit Location Theorem.*

Statement of Theorem:

Let $\{a_n\}$ *be a convergent sequence. Then*

$a_n \leq M$ *for* $n \gg 1 \Rightarrow \lim a_n \leq M$

$a_n \geq M$ *for* $n \gg 1 \Rightarrow \lim a_n \geq M$.

Proof: *We are given that* $a_n \leq M$ *for* $n \gg 1$ *and thatthere exists an* L *for which* $a_n \to L$.

Hence for all $\varepsilon > 0$, $a_n \underset{\varepsilon}{\approx} L$ *for* $n \gg 1$. *Equivalently,* $L - \varepsilon < a_n < L + \varepsilon$.

Since we are given that $a_n \leq M$ *for* $n \gg 1$, *it follows that* $L - \varepsilon < a_n \leq M$ *for* $n \gg 1$.

Now since ε is arbitrary, it follows that $L \leq M$.

5. For $n \geq 1$, define the sequence $\{b_n\}$ as follows:

$$b_n = \int_0^1 cos^n\left(\frac{\pi x}{2}\right)\ dx.$$

Prove that $b_n \to 0$.

Proof: *Let $\varepsilon > 0$ be given.*

Choose N^* such that $cos^{N^*}\left(\frac{\pi\varepsilon}{2}\right) < \varepsilon$. This can be achieved because

$0 < \cos\left(\frac{\pi\varepsilon}{2}\right) < 1$

Then $n > N^* \Rightarrow b_n = \int_0^1 cos^n\left(\frac{\pi x}{2}\right) dx = \int_0^{\varepsilon} cos^n\left(\frac{\pi x}{2}\right) dx + \int_{\varepsilon}^1 cos^n\left(\frac{\pi x}{2}\right) dx <$

$\int_0^{\varepsilon} cos^{N^*}\left(\frac{\pi x}{2}\right) dx + \int_{\varepsilon}^1 cos^{N^*}\left(\frac{\pi x}{2}\right) dx < (1)\,\varepsilon + \varepsilon(1-\varepsilon) < 2\varepsilon$.

(Here we have used the fact that $cos^n\left(\frac{\pi x}{2}\right)$ is decreasing on [0, 1].)

Hence by the K- ε principle, $b_n \to 0$.