**MATH 351 Solutions: TEST III 19 November 2018**



**Part I** *[5 points each]* ***Definitions & statements of theorems***

Be precise and careful.

1. State the *Intermediate Value Theorem*.
* *Let be continuous and assume that f(a) < f(b). Then*
1. State the *Maximum Theorem.*
* *Let be continuous. Then f(x) has a maximum and minimum on [a, b].*
1. State the *Squeeze Theorem* for functions.
* *Suppose that Then*
1. State the *Limit Location Theorem* for functions.
* *If the limits exist,*
1. Define *sequential compactness.*
* *A set is* ***sequentially compact*** *if every sequence of points in S has a subsequence converging to a point in S.*
1. State the *Positivity Theorem.*
* *If f is continuous at x = p, and f(p) > 0, then f(x) > 0 for*
1. Define *sequential continuity*.
* *F is* ***sequentially continuous*** *at x = p if given*

 *,*

**Part II** *[6 pts each]* ***Counter-Examples***

*Each of the following 9 assertions is false.* *Give an explicit counter-example to illustrate this.*

***Answer any 7 of the 9.*** *You may answer more than 7 for extra credit.*

1. If *H*: (0, 1) → **R** is continuous, then *H* is bounded.

*Counterexample:*

*Let H(x) = 1/x x*∈*(0, 1).*

1. Given two functions *f*: [0, 1] → **R** and*g*: [0, 1] → **R** such that each is *discontinuous* at x = 1/3, then is discontinuous at

*Counterexample:*

*Let*

*Then = 0 for all x*

1. Let *I* = (0, 1). If *g*: *I* → **R** is continuous, then *g*(*I*) is not a compact interval.

*Counterexample: Let .*

*Then g(I)=[3, 3].*

1. There does not exist a continuous function **R** that has neither a global maximum nor a global minimum on (0, 1] and does not have the limit ∞ or –∞ as

*Counterexample:*

*Note that for this example, x = 0 is an essential discontinuity.*

1. If two functions, *f*: **R** → **R** and*g*: **R** → **R** satisfy the conditions that

*Counterexample:*

*Let f(x) =*

1. There does not exist a function *f*: **R** → **R** that is continuous *only at x = 0.*

*Counterexample:*

*Let f(x) = x D(x) where D is the Dirichlet function defined by:*

1. If *F*: [0, 1] → **R** is continuous and {bn} is a sequence in [0, 1] for which {*F*(bn)} converges, then {bn} must converge.

*Counterexample:*

*Define F(x) = 5 for all x*∈*[0, 1]. Define*

1. If neither *S* nor *T* is a sequentially compact subset of **R**, then *is not* sequentially compact.

*Counterexample: Let S = [0, 3] and T = [2, 5].*

1. Let *h*: [0, 1] → **R** be a function that achieves a global maximum on [0, 1]. Then the

function F(x) = (h(x))4 also achieves a global maximum value on the interval [0, 1].

*Counterexample: Define*

*Note that h has a global maximum value of 0 on [0, 1] but that (h(x))4 has no global maximum on [0, 1].*

**Part III** *[12 pts each]* ***Proofs***

***Instructions:*** *Select* ***any 4 of the following 6 problems****. You may answer more than 4 to obtain extra credit.*

1. Using *only the definition of continuity*, prove that the function

 is continuous at x = 0.

*Solution: To begin, we conjecture that*

 *Let*

*Now since*

*Now let ; we find:*

*==*

*Hence f is continuous at x = 0.*

 *(Alternatively, one may use the K- principle.)*

1. *(a) Let p* Write *the negation* of the statement***f:* R → R is *continuous*** atx = p.

(*Note:* “*f* is *not continuous at p”* is not a sufficient answer. Use  and  in your answer.)

*Solution:*

*Since the definition of continuity of f at x = p is:*

*the logical negation of this sentence is:*

 *(b)* *Let* ***f:* R → R** and *p* Write *the negation* of the statement  *.*

*Solution: Since the definition of*  ***is***

,

*the logical negation of this sentence is:*

*.*

1. Let *f*: [0, 1] → [0, 1] be continuous. Prove that

*Hint:* Consider the two curves and on [0, 1]. Sketch a possible graph.

*Solution: Define h: [0, 1]* → ***R*** *as follows: h(x) = f(x) – x for all x∈[0, 1]. Now since f and the identity function are continuous, h must be continuous by the linearity theorem. Next note that h(0) = f(0) – 0 ≤ 0, by definition of f, and that h(1) = f(1) – 1 ≥ 0. Now, if either h(0) = 0 or h(1) = 0, then we are done, for it follows that either f(0) = 0 or f(1) = 1. So let us assume that h(0) < 0 and h(1) > 0. Then, invoking Bolzano’s Theorem, there exists p∈[a, b] such that h(p) = 0. And so, f(p) = p. (Note: p is called a* ***Fixed Point*** *of f.)*

1. A function *f*: **R** → **R** is said to be a *Lipschitz function* if there exists a constant L > 0 such that for all x, t ∈ ***R*** .

Prove that if *f:* **R** → **R** is a Lipschitz function then *f* is continuous at each p ∈ **R**.

*Hint:* Use the -definition of continuity.

*Solution: Let . Let be given.*

*We choose . Then*

*So f is continuous at x = p.*

1. Determine all values of the constants *A* and *B* so that the following function is *continuous*

*for all values of* *x*.

*Solution: If f is continuous at x = -1, then*

*Thus we obtain: -A – B = 2 – 3A + B.*

*Equivalently: A – B = 1 (\*)*

*If f is continuous at x = 1, then*

*Thus we obtain: 2 + 3A + B = 4;*

*Equivalently: 3A + B = 2. (\*\*)*

*Solving equations (\*) and (\*\*) simultaneously, we obtain:*

1. Prove that

*Solution: First observe that, since*

*Using the basic properties of the Riemann integral:*

*Next, since , the squeeze theorem yields the desired result.*