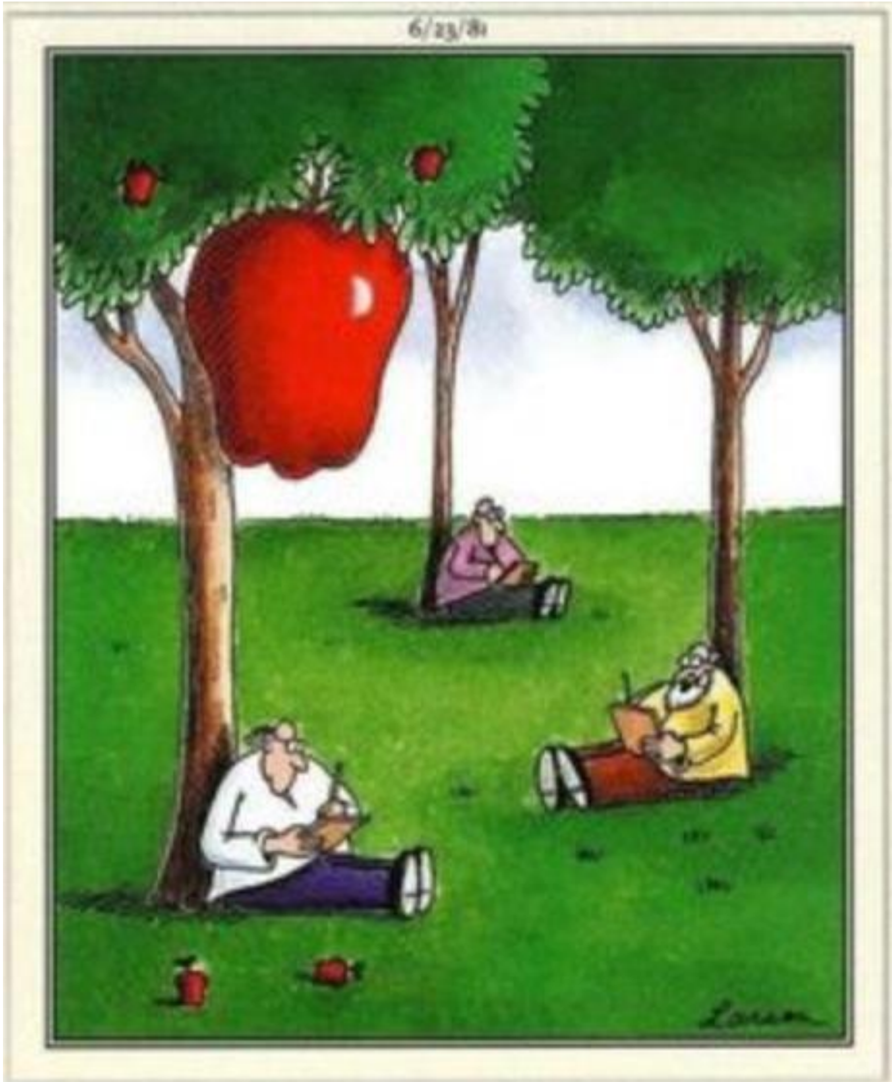
MATH 351 SOLUTIONS: TEST III 19 NOVEMBER 2018



"Nothing yet. ... How about you, Newton?"

PART I *[5 points each]* ***Definitions & statements of theorems***
Be precise and careful.

1. State the *Intermediate Value Theorem*.

- *Let* $f: [a, b] \to \mathbf{R}$ *be continuous and assume that f(a) < f(b). Then* $\forall z \in \mathbf{R}$,

$$f(a) \le z \le f(b) \ \Rightarrow \ \exists c \in [a, b] \text{ such that } f(c) = z.$$

2. State the *Maximum Theorem.*

- *Let* $f: [a, b] \to \mathbf{R}$ *be continuous. Then f(x) has a maximum and minimum on [a, b].*

3. State the *Squeeze Theorem* for functions.

- *Suppose that* $f(x) \le g(x) \le h(x)$ *for* $x \underset{\neq}{\approx} p$. *Then*

$$f(x) \to L \text{ and } h(x) \to L \text{ as } x \to p \ \Rightarrow g(x) \to L \text{ as } x \to p$$

4. State the *Limit Location Theorem* for functions.

- *If the limits exist,*

$$f(x) \leq M \; for \;\; x \underset{\neq}{\approx} p \;\; \Rightarrow \; \lim_{x \to p} f(x) \; \leq M$$

5. Define *sequential compactness.*

- *A set S ⊆ R is **sequentially compact** if every sequence of points in S has a subsequence converging to a point in S.*

6. State the *Positivity Theorem.*

- *If f is continuous at x = p, and f(p) > 0, then f(x) > 0 for $x \approx p$.*

7. Define *sequential continuity*.

- *F is **sequentially continuous** at x = p if given*

  $\{x_n\}, \; x_n \to p \;\; \Rightarrow f(x_n) \to f(p)$

PART II *[6 pts each]* ***Counter-Examples***

Each of the following 9 assertions is false. Give an explicit counter-example to illustrate this. ***Answer any 7 of the 9.*** *You may answer more than 7 for extra credit.*

1. If H: (0, 1) → **R** is continuous, then H is bounded.

Counterexample:

Let H(x) = 1/x ∀ $x \in (0, 1)$.

2. Given two functions f: [0, 1] → **R** and g: [0, 1] → **R** such that each is *discontinuous* at x = 1/3, then fg is discontinuous at $x = \frac{1}{3}$

Counterexample:

$$Let \;\; f(x) = \begin{cases} \frac{1}{3} & if \; x = 0 \\ 0 & if \; x \neq 0 \end{cases} \qquad Let \; g(x) = \begin{cases} 0 & if \; x = 0 \\ \frac{1}{3} & if \; x \neq 0 \end{cases}$$

Then $(fg)(x)$ *= 0 for all* $x \in [0, 1]$

3. Let $I = (0, 1)$. If $g: I \to \mathbf{R}$ is continuous, then $g(I)$ is not a compact interval.

Counterexample: Let $g(x) = 3 \quad \forall x \in (0,1)$.

Then g(I)=[3, 3].

4. There does not exist a continuous function $f: (0, 1] \to \mathbf{R}$ that has neither a global maximum nor a global minimum on (0, 1] and does not have the limit ∞ or $-\infty$ as $x \to 0^+$.

Counterexample: $$f(x) = \begin{cases} \frac{1}{x}\sin\frac{1}{x} & \text{if } x \in \left(0, \frac{1}{\pi}\right] \\ 0 & \text{if } x \in \left(\frac{1}{\pi}, 1\right] \end{cases}$$

Note that for this example, x = 0 is an essential discontinuity.

5. If two functions, $f: \mathbf{R} \to \mathbf{R}$ and $g: \mathbf{R} \to \mathbf{R}$ satisfy the conditions that

$$\lim_{x \to 0} g(x) = 13 \text{ and } \lim_{x \to 13} f(x) = 17$$

then $\lim_{x \to 0} f \circ g(x)$ exists and equals 17.

Counterexample:

Let $f(x) = \begin{cases} 17 & \text{if } x \neq 13 \\ 0 & \text{if } x = 13 \end{cases}$ and let $g(x) = \begin{cases} 13 & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$

6. There does not exist a function $f: \mathbf{R} \to \mathbf{R}$ that is continuous *only at x = 0.*

Counterexample:

Let f(x) = x D(x) where D is the Dirichlet function defined by:

$$D(x) = \begin{cases} 1 & \text{if } x \in Q \\ 0 & \text{if } x \notin Q \end{cases}$$

7. If $F: [0, 1] \to \mathbf{R}$ is continuous and $\{b_n\}$ is a sequence in [0, 1] for which $\{F(b_n)\}$ converges, then $\{b_n\}$ must converge.

Counterexample:

Define F(x) = 5 for all x∈[0, 1]. Define $b_n = \begin{cases} 1 & \text{if } n \in N \text{ is even} \\ 0 & \text{if } n \in N \text{ is odd} \end{cases}$

8. If neither S nor T is a sequentially compact subset of $\mathbf{R}$, then $S \cup T$ *is not* sequentially compact.

Counterexample: *Let S = [0, 3] and T = [2, 5].*

9. Let h: [0, 1] → **R** be a function that achieves a global maximum on [0, 1]. Then the function $F(x) = (h(x))^4$ also achieves a global maximum value on the interval [0, 1].

Counterexample: *Define* $h(x) = \begin{cases} -x & if\ 0 \le x < 1 \\ 0 & if\ x = 1 \end{cases}$

Note that h has a global maximum value of 0 on [0, 1] but that $(h(x))^4$ *has no global maximum on [0, 1].*

PART III *[12 pts each]* ***Proofs***

Instructions: *Select* ***any 4 of the following 6 problems****. You may answer more than 4 to obtain extra credit.*

1. Using *only the definition of continuity*, prove that the function

$$g(x) = \frac{x^4+5}{x^4+x+9}$$ is continuous at x = 0.

Solution: To begin, we conjecture that $g(x) \to \frac{5}{9}$ *as* $x \to 0$.

Let $\epsilon > 0$ *be given. Let* $\delta = min\left\{1,\ \frac{7}{13}\right\}$

Now since $|x| \le \delta \le 1$, *clearly* $|x| \le 1$ *and so*

$$|4x^3 - 5| \le 4|x^3| + 9| \le 13$$

Also

$$|x^4 + x + 9| \ge |9| - |x^4| - |x| \ge 9 - 2 = 7$$

Now let $\delta = min\{1, \frac{7}{13}\ \epsilon\}$*; we find:*

$$|g(x) - 1| = \left|\frac{x^4+5}{x^4+x+9} - \frac{5}{9}\right| = \left|\frac{9(x^4+5)-5(x^4+x+9)}{9(x^4+x+9)} - 1\right| =$$

$$\left|\frac{4x^4-5x}{x^4+x+9}\right| \le \frac{|4x^4-5x|}{7} = |x|\frac{|4x^3-5|}{7} \le \frac{13}{7}\ |x| < \frac{13}{7}\delta < \epsilon$$

Hence f is continuous at x = 0.

(Alternatively, one may use the K- ϵ principle.)

2. *(a)* *Let* $p \in \mathbf{R}$. Write *the negation* of the statement ***f*: R → R is *continuous*** at x = p.

(*Note:* "*f* is *not continuous at p*" is not a sufficient answer. Use ε and δ in your answer.)

Solution:

Since the definition of continuity of f at x = p is:

$$\forall \epsilon > 0\ \exists \delta > 0 \text{ such that } |f(x) - f(p)| < \varepsilon \text{ whenever } x \in \boldsymbol{R} \text{ and } |x - p| < \delta,$$

the logical negation of this sentence is:

$$\exists \epsilon > 0\ \ \forall \delta > 0\ \exists\, x \in \boldsymbol{R} \text{ such that} |f(x) - f(p)| \geq \varepsilon \text{ and } |x - p| < \delta.$$

(b) *Let* ***f*: R → R** and $p \in \mathbf{R}$. Write *the negation* of the statement $\lim_{x \to p} f(x) = \infty$.

Solution: *Since the definition of* $\lim_{x\to p} f(x) = \infty$ ***is***

$$\forall M > 0\ \exists \delta > 0 \text{ such that} |x - p| < \delta \ \Rightarrow\ f(x) > M,$$

the logical negation of this sentence is:

$$\exists M > 0\ \ \forall \delta > 0\ \exists x \text{ such that} |x - p| < \delta \text{ and } f(x) \leq M.$$

3. Let *f*: [0, 1] → [0, 1] be continuous. Prove that $\exists p \in [0, 1]$ such that $f(p) = p$.

Hint: Consider the two curves $y = f(x)$ and $y = x$ on [0, 1]. Sketch a possible graph.

Solution: *Define h: [0, 1] →* ***R*** *as follows: h(x) = f(x) – x for all x ∈[0, 1]. Now since f and the identity function are continuous, h must be continuous by the linearity theorem. Next note that h(0) = f(0) – 0 ≤ 0, by definition of f, and that h(1) = f(1) – 1 ≥ 0. Now, if either h(0) = 0 or h(1) = 0, then we are done, for it follows that either f(0) = 0 or f(1) = 1. So let us assume that h(0) < 0 and h(1) > 0. Then, invoking Bolzano's Theorem, there exists p ∈[a, b] such that h(p) = 0. And so, f(p) = p. (Note: p is called a* ***Fixed Point*** *of f.)*

4. A function *f*: **R** → **R** is said to be a *Lipschitz function* if there exists a constant L > 0 such that

for all x, t ∈ ***R*** $|f(x) - f(t)| < L|x - t|$.

Prove that if *f*: **R** → **R** is a Lipschitz function then *f* is continuous at each p ∈ **R**.

Hint: Use the (ε, δ)-definition of continuity.

Solution: *Let p* ∈ ***R*** . *Let* $\varepsilon > 0$ *be given.*

We choose $\delta = \frac{\varepsilon}{L}$. *Then*

$$|f(x) - f(p)| < L|x - p| < L\,\delta = \varepsilon$$

So f is continuous at x = p.

5. Determine all values of the constants *A* and *B* so that the following function is *continuous for all values of x*.

$$f(x) = \begin{cases} Ax - B & if\ x \le -1 \\ 2x^2 + 3Ax + B & if\ -1 < x \le 1 \\ 4 & if\ x > 1 \end{cases}$$

Solution: If f is continuous at x = -1, then $\lim_{x \to -1^+} f(x) = \lim_{x \to -1^-} f(x)$

Thus we obtain: -A – B = 2 – 3A + B.

Equivalently: A – B = 1 (*)

If f is continuous at x = 1, then $\lim_{x \to 1^+} f(x) = \lim_{x \to 1^-} f(x)$

Thus we obtain: 2 + 3A + B = 4;

Equivalently: 3A + B = 2. (**)

Solving equations (*) and (**) simultaneously, we obtain:

$$\boldsymbol{A = \frac{3}{4}}\ \ and\ \ \boldsymbol{B = -\frac{1}{4}}$$

6. Prove that $\lim_{x \to 0^+} \int_0^1 \frac{t^2}{1+t^4x} dt = \frac{1}{3}$.

Solution: *First observe that, since* $x \to 0^+$ *and* $0 \le t \le 1$

$$\frac{t^2}{1+x} \le \frac{t^2}{1+t^4x} \le t^2$$

Using the basic properties of the Riemann integral:

$$\frac{1}{1+x}\int_0^1 t^2\, dt = \int_0^1 \frac{t^2}{1+x} dt \le \int_0^1 \frac{t^2}{1+t^4x} dt \le \int_0^1 t^2\, dt = \frac{1}{3}$$

Next, since $\lim_{x \to 0^+} \frac{1}{1+x} \int_0^1 t^2\, dt = \int_0^1 t^2\, dt = \frac{1}{3}$, *the squeeze theorem yields the desired result.*