DISCUSSION QUESTIONS: 23 OCTOBER 2019

RELATED RATES

Review: **Implicit and logarithmic differentiation, inverse trig functions**

1. Differentiate each of the following functions:

(a) $y = \arcsin(3x)$

(b) $y = \arccos(5x - 13)$

(c) $y = (\text{arcsec } x) / x$

(d) $y = (\arctan x) + 3 (\arcsin x)$

(e) $y = \arctan((x - 1)/(x + 1))$

(f) $y = \ln(\ln x)$

(g) $y = \text{arc sin } (\text{arc tan } x)$

(h) $y = (\text{arc tan } x)^{3/2}$

(i) $y = (\ln(a + bx))^c$, where *a*, *b*, and *c* are constants

(j) (j) $y = g(x) = \cos(\ln x)$ Compute $g^{(100)}(x)$ and $g^{(101)}(x)$.

(k) $y = \text{arc tan } (1/x)$

2. (a) Let $y = u^3 + 1$ and $u = 5 \text{ arc sin } x$. Compute dy/dx

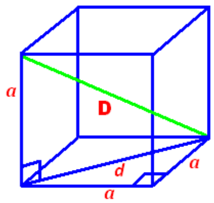
(b) Let $z = \arctan u$ and $u = e^x$. Compute dz/dx.

3. Prove the power rule for *rational* exponents, *viz.* $(d/dx)\, x^p = p\, x^{p-1}$ if *p* is rational.

Related Rates

1. If the original 10 cm edge length *x* of a cube decreases at the rate of 5 cm/min, when x = 4, at what rate does the cube's

(a) surface area change?



(b) volume change?

(c) main diagonal change?

2. Albertine (6 ft tall) is slowly walking away (at a rate of 20 ft/min) from a lamppost that is 10 ft tall. Determine the rate at which the length of her shadow changes at the moment when she is 12 ft from the base of the lamppost.

3. A cone-shaped coffee filter of radius 6 cm and depth 10 cm contains water, which drips out through a hole at the bottom at a constant rate of 1.5 cm^3 per second.

(a) If the filter starts out full, how long does it take to empty?

(b) Find the volume of water in the filter when the depth of the water is *h* cm.

(c) How fast is the water level falling when the depth is 8 cm?

4. A spherical snowball is melting. Its radius is decreasing at 0.2 cm per hour when the radius is 15 cm. How fast is its volume decreasing at that time?

5. A ruptured oil tanker causes a circular oil slick on the surface of the ocean. When its radius is 150 meters, the radius of the slick is expanding by 0.1 meter/minute, and its thickness is 0.02 meters. At that moment:

(a) How fast is the area of the slick expanding?

(b) The circular slick has the same thickness everywhere, and the volume of oil spilled remains fixed. How fast is the thickness of the slick decreasing?

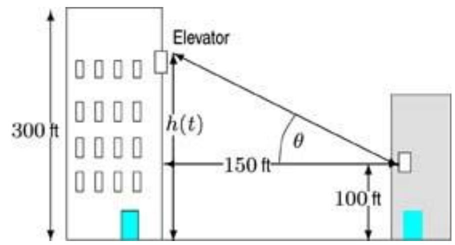
6. A gas station stands at the intersection of a north-south road and an east-west road. A police car is traveling toward the gas station from the east, chasing a stolen truck that is traveling north away from the gas station. The speed of the police car is 100 mph at the moment it is 3 miles from the gas station. At the same time, the truck is 4 miles from the gas station going 80 mph. At this moment:

(a) Is the distance between the car and truck increasing or decreasing? How fast? (Distance is measured along a straight line joining the car and the truck.)

(b) How does your answer change if the truck is traveling at a speed of 70 mph instead of 80 mph?

7. For the amusement of the guests, some hotels have elevators on the outside of the building. One such hotel is 300 feet high. You are standing by a window 100 feet above the ground and 150 feet away from the hotel, and the elevator descends at a constant speed of 30 ft/sec, starting at time

$t = 0$, where *t* is time in seconds. Let θ be the angle between the line of your horizon and your line of sight to the elevator. (See the figure below)



(a) Find a formula for h(t), the elevator's height above the ground as it descends from the top of the hotel.

(b) Using your answer to part (a), express θ as a function of time *t* and find the rate of change of θ with respect to *t*.

(c) The rate of change of θ is a measure of how fast the elevator appears to you to be moving. At what height is the elevator when it appears to be moving fastest?

8. Coroners estimate the time of death using the rule of thumb that a body cools about 2°F during the first hour after death and about 1°F for each additional hour. Assuming an air temperature of 68°F and a living body temperature of 98.6° F, the temperature T(*t*) in °F of a body at time *t* hours since death is given by

$T(t) = 68 + 30.6\, e^{-kt}$.

(a) For what value of *k* will the body cool by 2°F in the first hour?

(b) Using the value of *k* found in part (a), after how many hours will the temperature of the body be decreasing at a rate of 1°F per hour?

(c) Using the value of *k* found in part (a) show that, 24 hours after death, the coroner's rule of thumb gives approximately the same temperature as the formula.

9. (*U. of Michigan*) A certain type of spherical melon has weight proportional to its volume as it grows. When the melon weighs 0.2 pounds, it has a volume of 36 cm^{3}, and its weight is increasing at a rate of 0.1 pounds per day. (*Note:* The volume of a sphere of radius *r* is given by **$V = (4/3)\pi r^3$**.)

(a) Find dV/dt when the melon weighs 0.2 pounds (*t* is measured in days).

(b) Find the rate at which the radius of the melon is increasing when the melon weighs 0.2 pounds.

(c) Using a local linearization (i.e., tangent line approximation), estimate the volume of the melon 36 hours after it weighs 0.2 pounds.

STEWART EXERCISES

1. If V is the volume of a cube with edge length x and the cube expands as time passes, find dV/dt in terms of dx/dt.

   Answer

2.
   a. If A is the area of a circle with radius r and the circle expands as time passes, find dA/dt in terms of dr/dt.

   b. Suppose oil spills from a ruptured tanker and spreads in a circular pattern. If the radius of the oil spill increases at a constant rate of 1 m/s, how fast is the area of the spill increasing when the radius is 30 m?

3. Each side of a square is increasing at a rate of 6 cm/s. At what rate is the area of the square increasing when the area of the square is 16 cm^2?

   Answer

4. The length of a rectangle is increasing at a rate of 8 cm/s and its width is increasing at a rate of 3 cm/s. When the length is 20 cm and the width is 10 cm, how fast is the area of the rectangle increasing?

5. A cylindrical tank with radius 5 m is being filled with water at a rate of $3 \text{ m}^3/\text{min}$. How fast is the height of the water increasing?

   Answer

6. The radius of a sphere is increasing at a rate of 4 mm/s. How fast is the volume increasing when the diameter is 80 mm?

7. The radius of a spherical ball is increasing at a rate of 2 cm/min. At what rate is the surface area of the ball increasing when the radius is 8 cm?

8. The area of a triangle with sides of lengths a and b and contained angle θ is

$$A = \frac{1}{2}ab \sin \theta$$

   a. If $a = 2$ cm, $b = 3$ cm, and θ increases at a rate of 0.2 rad/min, how fast is the area increasing when $\theta = \pi/3$?

   b. If $a = 2$ cm, b increases at a rate of 1.5 cm/min, and θ increases at a rate of 0.2 rad/min, how fast is the area increasing when $b = 3$ cm and $\theta = \pi/3$?

   c. If a increases at a rate of 2.5 cm/min, b increases at a rate of 1.5 cm/min, and θ increases at a rate of 0.2 rad/min, how fast is the area increasing when $a = 2$ cm, $b = 3$ cm, and $\theta = \pi/3$?

9. Suppose $y = \sqrt{2x+1}$, where x and y are functions of t.

   a. If $dx/dt = 3$, find dy/dt when $x = 4$.

   Answer ↓

   b. If $dy/dt = 5$, find dx/dt when $x = 12$.

   Answer ↓

10. Suppose $4x^2 + 9y^2 = 36$, where x and y are functions of t.

   a. If $dy/dt = \frac{1}{3}$, find dx/dt when $x = 2$ and $y = \frac{2}{3}\sqrt{5}$.

   b. If $dx/dt = 3$, find dy/dt when $x = -2$ and $y = \frac{2}{3}\sqrt{5}$.

11. If $x^2 + y^2 + z^2 = 9$, $dx/dt = 5$, and $dy/dt = 4$, find dz/dt when $(x, y, z) = (2, 2, 1)$.

12. A particle is moving along a hyperbola $xy = 8$. As it reaches the point $(4, 2)$, the y-coordinate is decreasing at a rate of 3 cm/s. How fast is the x-coordinate of the point changing at that instant?

13, 14, 15 and 16

a. What quantities are given in the problem?

b. What is the unknown?

c. Draw a picture of the situation for any time t.

d. Write an equation that relates the quantities.

e. Finish solving the problem.

13. A plane flying horizontally at an altitude of 1 mi and a speed of 500 mi/h passes directly over a radar station. Find the rate at which the distance from the plane to the station is increasing when it is 2 mi away from the station.

14. If a snowball melts so that its surface area decreases at a rate of $1\ \text{cm}^2/\text{min}$, find the rate at which the diameter decreases when the diameter is 10 cm.

15. A street light is mounted at the top of a 15-ft-tall pole. A man 6 ft tall walks away from the pole with a speed of 5 ft/s along a straight path. How fast is the tip of his shadow moving when he is 40 ft from the pole?

16. At noon, ship A is 150 km west of ship B. Ship A is sailing east at 35 km/h and ship B is sailing north at 25 km/h. How fast is the distance between the ships changing at 4:00 PM?

17. Two cars start moving from the same point. One travels south at 60 mi/h and the other travels west at 25 mi/h. At what rate is the distance between the cars increasing two hours later?

Answer

18. A spotlight on the ground shines on a wall 12 m away. If a man 2 m tall walks from the spotlight toward the building at a speed of 1.6 m/s, how fast is the length of his shadow on the building decreasing when he is 4 m from the building?

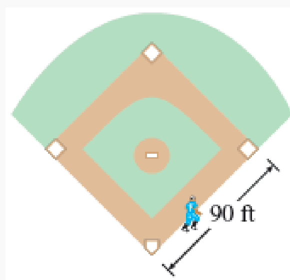
19. A man starts walking north at 4 ft/s from a point P. Five minutes later a woman starts walking south at 5 ft/s from a point 500 ft due east of P. At what rate are the people moving apart 15 min after the woman starts walking?

Answer

20. A baseball diamond is a square with side 90 ft. A batter hits the ball and runs toward first base with a speed of 24 ft/s.

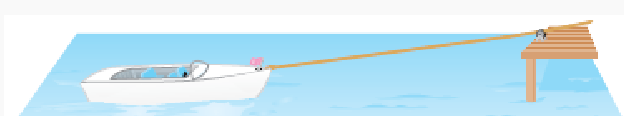
a. At what rate is his distance from second base decreasing when he is halfway to first base?

b. At what rate is his distance from third base increasing at the same moment?



21. The altitude of a triangle is increasing at a rate of 1 cm/min while the area of the triangle is increasing at a rate of $2 \text{ cm}^2\text{/min}$. At what rate is the base of the triangle changing when the altitude is 10 cm and the area is 100 cm^2?

22. A boat is pulled into a dock by a rope attached to the bow of the boat and passing through a pulley on the dock that is 1 m higher than the bow of the boat. If the rope is pulled in at a rate of 1 m/s, how fast is the boat approaching the dock when it is 8 m from the dock?



23. At noon, ship A is 100 km west of ship B. Ship A is sailing south at 35 km/h and ship B is sailing north at 25 km/h. How fast is the distance between the ships changing at 4:00 PM?

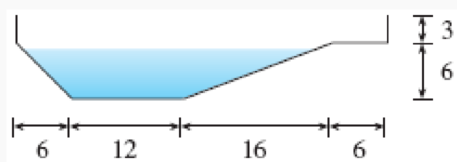
24. A particle moves along the curve $y = 2\sin(\pi x/2)$. As the particle passes through the point $\left(\frac{1}{3}, 1\right)$, its x-coordinate increases at a rate of $\sqrt{10}$ cm/s. How fast is the distance from the particle to the origin changing at this instant?

25. Water is leaking out of an inverted conical tank at a rate of $10{,}000\ \mathrm{cm}^3/\mathrm{min}$ at the same time that water is being pumped into the tank at a constant rate. The tank has height 6 m and the diameter at the top is 4 m. If the water level is rising at a rate of 20 cm/min when the height of the water is 2 m, find the rate at which water is being pumped into the tank.

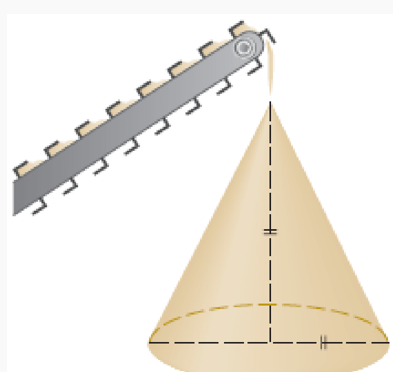
26. A trough is 10 ft long and its ends have the shape of isosceles triangles that are 3 ft across at the top and have a height of 1 ft. If the trough is being filled with water at a rate of $12\ \mathrm{ft}^3/\mathrm{min}$, how fast is the water level rising when the water is 6 inches deep?

27. A water trough is 10 m long and a cross-section has the shape of an isosceles trapezoid that is 30 cm wide at the bottom, 80 cm wide at the top, and has height 50 cm. If the trough is being filled with water at the rate of $0.2\ \mathrm{m}^3/\mathrm{min}$, how fast is the water level rising when the water is 30 cm deep?

28. A swimming pool is 20 ft wide, 40 ft long, 3 ft deep at the shallow end, and 9 ft deep at its deepest point. A cross-section is shown in the figure. If the pool is being filled at a rate of $0.8\ \mathrm{ft}^3/\mathrm{min}$, how fast is the water level rising when the depth at the deepest point is 5 ft?



29. Gravel is being dumped from a conveyor belt at a rate of $30\ \mathrm{ft}^3/\mathrm{min}$, and its coarseness is such that it forms a pile in the shape of a cone whose base diameter and height are always equal. How fast is the height of the pile increasing when the pile is 10 ft high?



30. A kite 100 ft above the ground moves horizontally at a speed of 8 ft/s. At what rate is the angle between the string and the horizontal decreasing when 200 ft of string has been let out?

31. The sides of an equilateral triangle are increasing at a rate of 10 cm/min. At what rate is the area of the triangle increasing when the sides are 30 cm long?

Answer

32. How fast is the angle between the ladder and the ground changing in Example 2 when the bottom of the ladder is 6 ft from the wall?

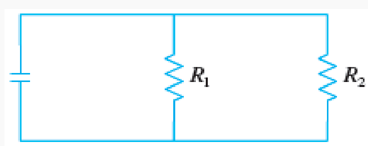
33. The top of a ladder slides down a vertical wall at a rate of 0.15 m/s. At the moment when the bottom of the ladder is 3 m from the wall, it slides away from the wall at a rate of 0.2 m/s. How long is the ladder?

38. When air expands adiabatically (without gaining or losing heat), its pressure P and volume V are related by the equation $PV^{1.4} = C$, where C is a constant. Suppose that at a certain instant the volume is 400 cm^3 and the pressure is 80 kPa and is decreasing at a rate of 10 kPa/min. At what rate is the volume increasing at this instant?

39. If two resistors with resistances R_1 and R_2 are connected in parallel, as in the figure, then the total resistance R, measured in ohms (Ω), is given by

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}$$

If R_1 and R_2 are increasing at rates of 0.3 Ω/s and 0.2 Ω/s, respectively, how fast is R changing when $R_1 = 80\ \Omega$ and $R_2 = 100\ \Omega$?



44. A lighthouse is located on a small island 3 km away from the nearest point P on a straight shoreline and its light makes four revolutions per minute. How fast is the beam of light moving along the shoreline when it is 1 km from P?

45. A plane flies horizontally at an altitude of 5 km and passes directly over a tracking telescope on the ground. When the angle of elevation is $\pi/3$, this angle is decreasing at a rate of $\pi/6$ rad/min. How fast is the plane traveling at that time?

Answer

46. A Ferris wheel with a radius of 10 m is rotating at a rate of one revolution every 2 minutes. How fast is a rider rising when his seat is 16 m above ground level?

47. A plane flying with a constant speed of 300 km/h passes over a ground radar station at an altitude of 1 km and climbs at an angle of 30°. At what rate is the distance from the plane to the radar station increasing a minute later?

Answer

48. Two people start from the same point. One walks east at 3 mi/h and the other walks northeast at 2 mi/h. How fast is the distance between the people changing after 15 minutes?

49. A runner sprints around a circular track of radius 100 m at a constant speed of 7 m/s. The runner's friend is standing at a distance 200 m from the center of the track. How fast is the distance between the friends changing when the distance between them is 200 m?

Answer

50. The minute hand on a watch is 8 mm long and the hour hand is 4 mm long. How fast is the distance between the tips of the hands changing at one o'clock?