Math 161 6 November 2019

Optimization review probs *(Univ Michigan)*

1. Eddie and Laura have signed an exclusive contract to begin producing the world's first caffeinated soup, called **Minestromnia**. If they charge \$4.00 per liter or more for the soup, then nobody will buy it. Otherwise, if they charge *p* dollars per liter for the soup, they will sell g(p) liters, where

$$g(p) = 500(16 - p^2).$$

a. *[3 points]* Write an expression for the revenue R(p) that Eddie and Laura will generate if they charge p dollars per liter of soup.

b. *[3 points]* The ingredients in a liter of Minestromnia cost \$1.00. To start their business, Eddie and Laura need to purchase a huge soup kettle and other equipment at a total cost of \$700.00. Write an expression for the total cost C(p), including fixed costs, of producing g(p) liters of soup.

c. *[6 points]* What price should Eddie and Laura charge per liter of Minestromnia to maximize their profits? Be sure to explain how you know that this price produces the maximum possible profit.

d. *[4 points]* Give a practical interpretation of the formula $g'(3.5) = -3500$ that begins with "If Eddie and Laura decrease the price of the soup from \$3.50 per liter to \$3.40 per liter . . .

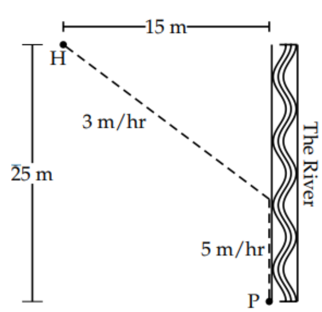
2. *[10 points]* As a software engineer, Tendai spends many hours every day **writing code**. Let w(t) be a function that models the number of lines of code that Tendai writes in a day if he works *t* hours that day. Tendai works at least one hour and at most 18 hours each day. A formula for w(t) is given by

$$w(t) = \begin{cases} -2t^2 + 28t & \text{if } 1 \le t \le 3 \\ -0.5t^2 + 9t + 43.5 & \text{if } 3 < t \le 18. \end{cases}$$

a. *[8 points]* Find the values of *t* (if they exist) that minimize and maximize w(t) on the interval [1, 18]. Use calculus to find your answers, and be sure to show enough evidence that the points you find are indeed global extrema.

b. *[2 points]* What is the largest number of lines of code that Tendai can expect to write in a day according to this model?

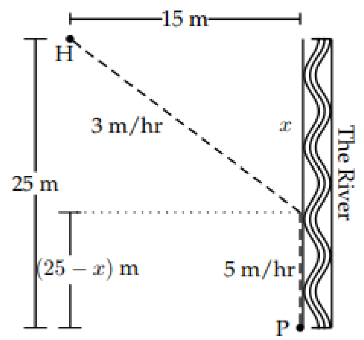
3. *[16 points]* The **Awkward Turtle** is going to a dinner party! Unfortunately, he's running quite late, so he wants to take the quickest route. The Awkward Turtle lives in a grassy plain (his home is labeled H in the figure below), where his walking speed is a slow but steady 3 meters per hour. The party is taking place southeast of his home, on the bank of a river (denoted by *P* in the figure). The river flows south at a constant rate of 5 meters per hour, and once he gets to the river, the Awkward Turtle can jump in and float the rest of the way to the party on his back. A typical path the Awkward Turtle might take from his house to the party is indicated in the figure below by a dashed line. What is the shortest amount of time the entire trip (from home to dinner party) can take? [Recall that rate × time = distance.]



Awkward Turtle solution

(16 points) The Awkward Turtle is going to a dinner party! Unfortunately, he's running quite late, so he wants to take the quickest route. The Awkward Turtle lives in a grassy plain (his home is labeled H in the figure below), where his walking speed is a slow but steady 3 meters per hour. The party is taking place southeast of his home, on the bank of a river (denoted by P in the figure). The river flows south at a constant rate of 5 meters per hour, and once he gets to the river, the Awkward Turtle can jump in and float the rest of the way to the party on his back. A typical path the Awkward Turtle might take from his house to the party is indicated in the figure below by a dashed line.

What is the shortest amount of time the entire trip (from home to dinner party) can take? [*Recall that rate × time = distance.*]



Let $t(x)$ denote the amount of time the trip takes if the Awkward Turtle floats along the river for $(25 - x)$ meters (see the picture above). By the Pythagorean theorem, the distance the turtle will walk across the grassy plain is $\sqrt{15^2 + x^2}$; therefore, we have

$$t(x) = \frac{1}{3}\sqrt{15^2 + x^2} + \frac{25 - x}{5}.$$

To minimize this, we take the derivative and set it equal to 0. A computation shows that

$$t'(x) = \frac{1}{3} \cdot \frac{x}{\sqrt{15^2 + x^2}} - \frac{1}{5}.$$

Setting this equal to 0 and solving yields $x = \frac{45}{4} = 11.25$ m. Thus, such a trip takes $t(11.25) = 9$ hours.

Using the quotient rule we see that

$$t''(x) = \frac{1}{3}\left(\frac{\sqrt{15^2 + x^2} - \frac{x^2}{\sqrt{15^2+x^2}}}{15^2 + x^2}\right) = \frac{1}{3}\left(\frac{15^2}{\left(\sqrt{15^2 + x^2}\right)^3}\right)$$

Since $t''(x) > 0$ for all x, we have that 9 hours is a local minimum. To determine whether $t = 9$ is the global minimum, we can either show that since there is only one critical point the local minimum is the global minimum or check the endpoints. The least the turtle can float along the river is 0 meters, in which case the trip takes $t(0) = 10$ hours; the greatest distance he can float is 25 meters, in which case the trip would take $t(25) \approx 9.718$ hours. Therefore, 9 hours is indeed the global minimum. (Alternatively, one can see this from the graph the function $y = t(x)$.)

Minimal time = 9 hours

Coding Output solution

[10 points] As a software engineer, Tendai spends many hours every day writing code. Let $w(t)$ be a function that models the number of lines of code that Tendai writes in a day if he works t hours that day. Tendai works at least one hour and at most 18 hours each day. A formula for $w(t)$ is given by

$$w(t) = \begin{cases} -2t^2 + 28t & \text{if } 1 \le t \le 3 \\ -0.5t^2 + 9t + 43.5 & \text{if } 3 < t \le 18. \end{cases}$$

a. [8 points] Find the values of t that minimize and maximize $w(t)$ on the interval $[1, 18]$. Use calculus to find your answers, and be sure to show enough evidence that the points you find are indeed global extrema. For each answer blank, write NONE if appropriate.

Solution: Note that w is continuous at $t = 3$, since $\lim_{t\to 3^-} w(t) = \lim_{t\to 3^+} w(t) = 66$, so we may use the Extreme Value Theorem.
We find

$$w'(t) = \begin{cases} -4t + 28 & \text{if } 1 < t < 3 \\ -t + 9 & \text{if } 3 < t < 18. \end{cases}$$

The first expression is 0 when $t = 7$, but since this isn't in the domain of that piece, it is not a critical point. The second expression is 0 when $t = 9$.
Since both of these are polynomials, we don't have to worry about the derivative not existing on these open intervals. However, since $-4 \cdot 3 + 28 = 16$ and $-3 + 9 = 6$ are not equal, w' is not defined at 3, so $t = 3$ is also a critical point.

Computing $w(t)$ at each critical point and the endpoints gives:

t	1	3	9	18
$w(t)$	26	66	84	43.5

By the Extreme Value Theorem, we therefore find that $w(t)$ attains its maximum value at $t = 9$ and its minimum at $t = 1$.

Answer: global max(es) at $t =$ 9

Answer: global min(s) at $t =$ 1

b. [2 points] What is the largest number of lines of code that Tendai can expect to write in a day according to this model?

Solution: From part **a.** we see that the maximum value of w is $w(9) = 84$. So according to this model, the largest number of lines of codes that Tendai can expect to write in a day is 84.

Minestromnia solution

. [16 points] Eddie and Laura have signed an exclusive contract to begin producing the world's first caffeinated soup, called Minestromnia. If they charge \$4.00 per liter or more for the soup, then nobody will buy it. Otherwise, if they charge p dollars per liter for the soup, they will sell $g(p)$ liters, where

$$g(p) = 500(16 - p^2).$$

a. [3 points] Write an expression for the revenue $R(p)$ that Eddie and Laura will generate if they charge p dollars per liter of soup.

Solution: The revenue is the price times the number of liters sold, so

$$R(p) = pg(p) = 500p(16 - p^2)$$

(To be really precise, we should say

$$R(p) = \begin{cases} 500p(16 - p^2), & 0 \leq p < 4 \\ 0, & p \geq 4 \end{cases}$$

or something like this.)

b. [3 points] The ingredients in a liter of Minestromnia cost \$1.00. To start their business, Eddie and Laura need to purchase a very large soup kettle and other equipment at a total cost of \$700.00. Write an expression for the total cost $C(p)$, including fixed costs, of producing $g(p)$ liters of soup.

Solution: The fixed costs are 700 and each liter costs a dollar, for a total cost of

$$C(p) = 700 + g(p) = 700 + 500(16 - p^2).$$

c. [6 points] What price should Eddie and Laura charge per liter of Minestromnia in order to maximize their profits? Be sure to explain how you know that this price produces the maximum possible profit.

Solution: Write $\pi(p) = R(p) - C(p)$. Then after some algebra, we get $\pi(p) = -500p^3 + 500p^2 + 8000p - 8700$, so

$$\pi'(p) = -1500p^2 + 1000p + 8000$$

Setting $\pi'(p)$ equal to zero gives a quadratic equation, whose solutions are $2\frac{2}{3} \approx 2.67$ and the illogical -2. So the critical point is $p = 2.67$. Plugging in the critical point gives

$$\pi(2.67) \approx 6707.$$

The reasonable prices that Eddie and Laura can set lie in a closed interval: $0 \leq p \leq 5$. The profit is clearly negative at both endpoints (they aren't getting any revenue at either endpoint) so the maximum profit occurs when the price is approximately \$2.67.

d. [4 points] Give a practical interpretation of the formula

$$g'(3.5) = -3500$$

that begins with
"If Eddie and Laura decrease the price of the soup from \$3.50 per liter to \$3.40 per liter . . ."

Solution: . . . they expect to sell about 350 more liters of Minestromnia.