MATH 161: PRACTICE PROBLEMS FOR TEST I (REVISED)

25 SEPTEMBER 2019

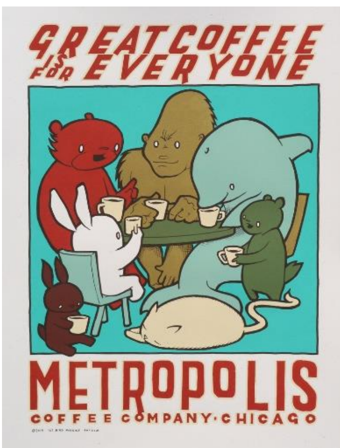
1) Using *only the limit definition of the derivative*, write $g'(5)$ if $g(x) = x^{\left(\frac{1-2x}{1+3x}\right)}$. Do not attempt to evaluate!

2) (a) Using only the limit definition of the derivative compute the value of $f'(1)$ if $f(x) = \frac{x}{x+1}$

(b) Using short cuts compute $f'(1)$.

3) Albertine orders a large cup of coffee at Metropolis on Granville. Let F(t) be the temperature in *degrees Fahrenheit* of her coffee *t minutes* after the coffee is placed on her tray.

(a) Explain the meaning of the statement: F(9) = 167. (Use complete sentences. Avoid any mathematical terms!)



(b) Explain the meaning of the statement: $F^{-1}(99) = 17.5$

(c) Give the *practical* interpretation of the statement: $F'(9) = -1.10$. (Use complete sentences. Do not use the words "derivative" or "rate" or any other mathematical term in your explanation.)

(d) What are the *units* of $F'(9)$?

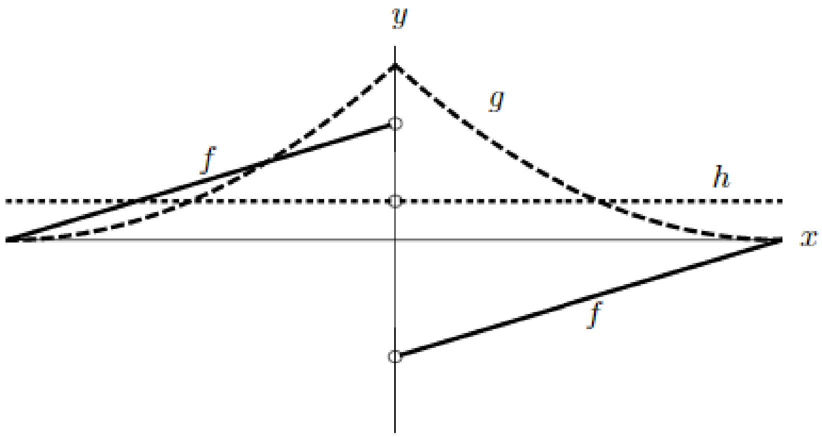
(e) Using the information given in parts (a) and (c), estimate the temperature of Albertine's coffee *seven* minutes after she has been handed the coffee.

(f) [EXTRA CREDIT] Explain the meaning of the statement:

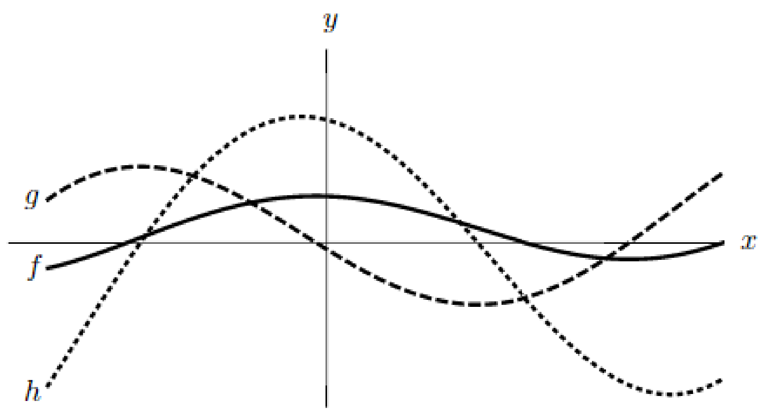
$$(F^{-1})'(99) = -1$$

4) For each of the following three sets of axes, exactly one of the following statements (a) – (e) is true. You may use a letter more than once. In the space provided next to each figure, enter the letter of the true statement for that figure. For each graph, note that the x and y scales are not the same.

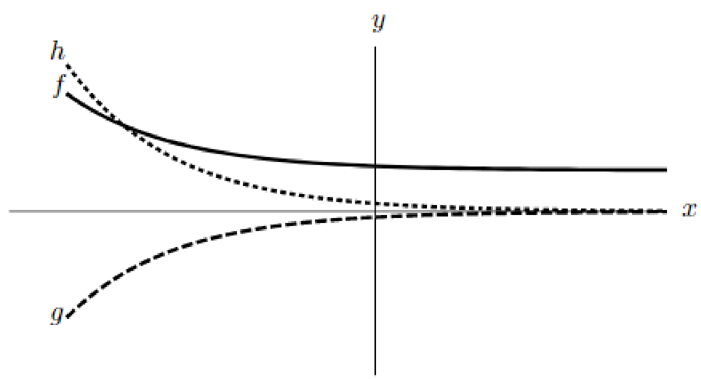
(a) h is the derivative of f, and f is the derivative of g.
(b) g is the derivative of f, and f is the derivative of h.
(c) g is the derivative of h, and h is the derivative of f.
(d) h is the derivative of g, and g is the derivative of f.
(e) None of (a)-(d) are possible.



True statement _____



True statement _____



True statement _____

5) Using only the limit definition of the derivative, write an explicit expression for the *derivative* of the function

$g(x) = (\cos x)^x$ at $x = 3$. *Do not try to calculate this derivative.*

6) *(a) Find* $\lim_{x\to\infty} f(x)$ *if, for all* $x > 5$,

$$\frac{4x-1}{x} < f(x) < \frac{4x^2+3x}{x^2}$$

Explain! Which theorem are you using?

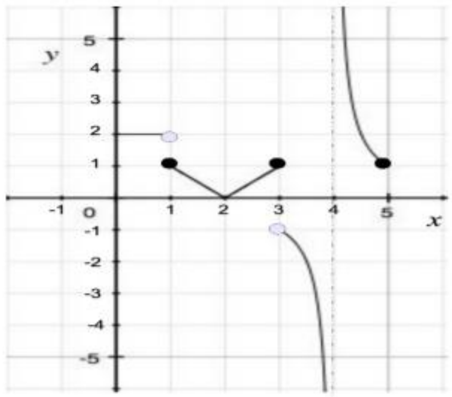
(b) Show that y = f(x) = $x^3 + 5e^x + 1$ has *at least one* real root. *Explain!*

Which theorem are you using?

7) Find an equation of the *normal line* to the curve

$y = g(x) = \frac{x^2-1}{x^2+1}$ at x = 1.
You may use short cuts.

8) The graph of a function *g* is shown below. For each of the following, decide if the limit exists. If it does, find the limit. If it does not, decide also if the "limit" is ∞, -∞, or neither. No justification is necessary for full credit, but show your work for purposes of partial credit.



(a) $\lim_{x\to 0^+} g(x) =$ *(b)* $\lim_{x\to 1^-} g(x)=$ *(c)* $\lim_{x\to 1^+} g(x)=$ *(d)* $\lim_{x\to 1} g(x)=$ *(e)* $\lim_{x\to 2} g(x)=$

(f) $\lim_{x\to 0^+} g(x) =$ *(g)* $\lim_{x\to 3^-} g(x)=$ *(h)* $\lim_{x\to 3^+} g(x) =$ *(i)* $\lim_{x\to 3} g(x) =$

(j) $\lim_{x\to 4^+} g(x) =$ *(k)* $\lim_{x\to 4^-} g(x) =$ *(l)* $\lim_{x\to 4} g(x) =$ *(m)* $\lim_{x\to 5^-} g(x) =$

9) For each of the following functions, determine the type of discontinuity at the given point. If it is a *removable* discontinuity, find continuous extension of the function.

(a) $y = \frac{x^3-x^2-2x}{(x-2)(x+5)}$ at x = 2

(b) $y = \frac{x^3-x^2-2x}{(x-2)(x+5)}$ at x = -5

(c) $y = \cos\frac{3}{x}$ $at\ x = 0$

(d) $y = \frac{|x|}{x}$ $at\ x = 0$

10) Suppose that *f* and *g* are differentiable functions satisfying:

f(3) = -2, g(3) = -4, $f'(3) = 3$, and $g'(3) = -1$.

(a) Let $H(x) = (f(x) + 2g(x) + 1)(f(x) - g(x) - 4)$. Compute $H'(3)$ (Hint: Use short cuts here.)

(b) Let $M(x) = \dfrac{2f(x) + 3g(x)}{2 - 3g(x)}$. Compute $M'(3)$

11) For each of the following, find any and all critical points. Then, using the first derivative test, classify them (local max, local min, neither).

(a) $y = x^3 - 3x + 1$

(b) $y = 3x^4 - 16x^3 + 18x^2 + 1$

12) Let y = f(x) be a differentiable function with derivative

$$f'(x) = \frac{e^x(x-1)(x-2)^2(x-4)^3(x-5)^4(x-6)^5}{1+x^4}$$

(a) Find any and all critical points.

(b) Classify each critical point (local max, local min, neither).

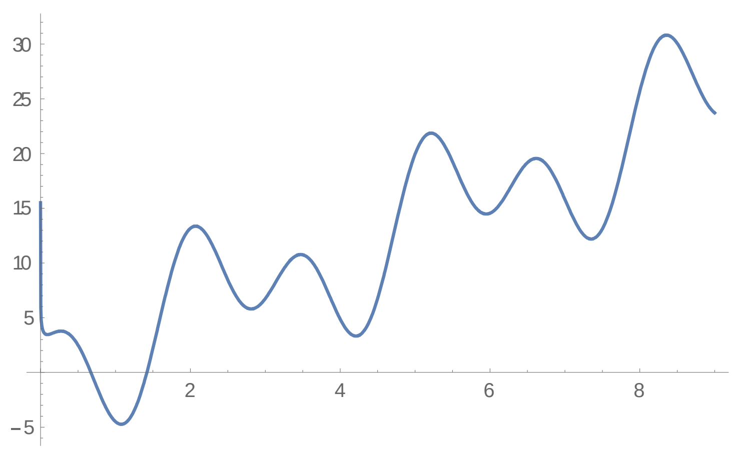
13) Compute each of the following limits. *Justify your reasoning.*

(a) $\lim\limits_{x\to\infty} \dfrac{(4x^3+11)^2(3x-91)^3}{(2x^2+5)^4(2x+2017)}$ (b) $\lim\limits_{x\to 3} \dfrac{\frac{1}{x^2} - \frac{1}{9}}{x-3}$

(c) $\lim\limits_{x\to\infty} \dfrac{\sin 7x}{x}$ (d) $\lim\limits_{x\to 0} \dfrac{\sqrt{x+1} - 1}{x}$

14) Using the process of "geometric differentiation," sketch the graph of the derivative of the function y = G(x) whose graph is given below.

FYI: This is the graph of $y = f(x) = 9\sin x \cos 3x + 3x - \ln x$



15) (a) State carefully the *Intermediate Value Theorem.*

(b) Using the Intermediate Value Theorem, prove that the polynomial function $g(x) = x^4 - 7x^2 + x + 5$ has at least one real negative root x.

16) Sketch a continuous, differentiable graph with the following properties:

- local minima at 2 and 4
- global minimum at 2
- local and global maximum at 3
- no other extrema

17) Let $f(x) = x^4 - a\,x^2$.

(a) Find all possible critical points of *f* in terms of *a*.

(b) If $a < 0$, how many critical points does *f* have?

(c) If $a > 0$, find the *x* and *y* coordinates of all critical points of *f*.

18) Given $f(x) = x^6 - 3x^5$ on the interval [-1, 4].

(a) Find all critical points of *f*.

(b) Determine on which intervals *f* is increasing.

(c) Find and classify all local and global extrema of *f*.

(d) Sketch the graph of *f* using the above information.

19) Given the function $f(x) = x\ln(2x) - x$ on the closed interval $[1/(2e),\ e/2]$, find the global extrema and use this information to sketch the graph. Identify all local and global extrema. *Sketch.*

20) *[University of Michigan]* A model for the amount of an antihistamine in the bloodstream after a patient takes a dose of the drug gives the amount, a, as a function of time, t, to be

$a(t) = A(e^{-t} - e^{-kt})$. In this equation, *A* is a measure of the dose of antihistamine given to the patient, and *k* is a transfer rate between the gastrointestinal tract and the bloodstream. *A* and *k* are positive constants, and for pharmaceuticals such as antihistamine, $k > 1$.

(a) Find the location $t = T_m$ of the non-zero critical point of a(t).

(b) Explain why $t = T_m$ is a global maximum of a(t) by referring to the expression for a(t) or

21). The derivative of a continuous function g is given by

$$g'(x) = \frac{e^{-5x}(x+2)(x-3)^2(x^4+3)(x-9)}{(x-5)^{1/3}}$$

Determine all critical points of g, and classify each as a local max, local min, or neither. Carefully explain your reasoning for each classification.

22) Using an appropriate tangent line approximation, estimate the value of each of the following quantities:

(a) $(1.00013)^{1/9}$

(b) $(0.99999)^5$

(c) $e^{0.0007}$

(d) $48.993^{1/2}$

23) Given $f(x) = x^4 - 4x^3 - 8x^2 + 1$ on the interval [-5, 5].

(a) Find all critical points of *f*.

(b) Determine on which intervals *f* is increasing.

(c) Using the information obtained above, sketch the graph of y = f(x).

24) Given $f(x) = x(x - 2)^4$ on the real line.

(a) Find all critical points of *f*.

(b) Determine on which intervals *f* is increasing.

(c) Sketch the graph of *f* using the above information.

25) Given H(x) = x + 2 sin x on the interval [0, 4π].

(a) Find all critical points of *H*.

(b) Determine on which intervals *H* is increasing.

26) Given $G(x) = x^2 / (x^2 + 3)$ on the real line.

(a) Find all critical points of *G*.

(b) Determine on which intervals *G* is increasing.

(c) Sketch the graph of *G* using the above information.

27) For each of the following functions, determine *where* the tangent line is horizontal. (The x-coordinates of these critical points are sufficient.)

Here, since we haven't yet studied, the Chain Rule, you may need to be given formulae for differentiating $y = (ax + b)^n$ and $y = e^{cx}$.

$$(a)\quad y = (3x-1)^8(2x+1)^{13}$$

$$(b)\quad y = \frac{(4x+3)^{31}}{(x-2)^{15}}$$

$$(c)\quad y = (x+1)^4 e^{3x}$$

28) For each of the following curves, determine all local extrema. $y = xe^{2x}$

(a) $y = x^3/3 - x^2/2 - 2x + 1$

(b) $y = x + \sin x$ on the interval $[-2\pi/3, 2\pi/3]$

(c) $y = x(1 - x)^2$

(d) $y = x^2(x^2 - 2)$

29) For which value or values of the constant k will the curve

$$y = x^3 + kx^2 + 3x - 4$$

have *exactly one* horizontal tangent?

30) Sketch the graph of the function $g(x) = x^2(x - 1)^2$ on the interval [0, 2]. Locate any (and all) local and global extrema.

31) Suppose that the derivative of the function

$y = f(x)$ is $y' = (x - 1)^2(x - 2)$.

Find and classify all local extrema.

32) Suppose that the derivative of the function $y = g(x)$ is

$$y' = x^2(x - 2)^3(x + 3).$$

Find and classify all local extrema.

33) Find the values of constants *a*, *b*, and *c* so that the graph of

$y = (x^2 + a) / (bx + c)$ has a local *minimum* at $x = 3$ and a local *maximum* at (-1, -2).

34) The quantity, *Q* mg, of nicotine in the body *t* minutes after a cigarette is smoked is given by $Q = g(t)$.

(a) Using a *complete sentence*, interpret the statement $g(20) = 0.36$ *without using any mathematical terminology.*

(b) What are the units of dg/dt ?

(c) Using a *complete sentence*, interpret the statement $g'(20) = -0.002$ *without using any mathematical terminology.*

(d) *[1 pt]* Using the information that you obtained above, *estimate* g(23). *(As usual, show your work!)*

35) A scientist is growing a very large quantity of mold. Initially, the mass of mold grows exponentially, but after many hours, the mass stabilizes at 24 kilograms.

Suppose that *t* hours after the scientist begins, the mass of mold, in kilograms, can be modeled by the function M defined by the equation

$$M(t) = \begin{cases} 0.41e^{0.72t} & \text{if } 0 \le t \le 5 \\ \dfrac{2t^3}{at^b + c} & \text{if } t > 5. \end{cases}$$

a. Find the value of k between 0 and 5 so that M(k) = 1. Then interpret the equation M(k) = 1 in the context of this problem. Use a complete sentence and include units.

b. Assuming that M is a continuous function of t, determine $\lim_{x\to\infty} M(t)$ and the values of a, b, and c.

36) Find $\lim_{x\to 0} \sin\left(\frac{1}{\ln|x|}\right)$

DERIVATIVE RULES

$\frac{d}{dx}\left(x^n\right) = nx^{n-1}$

$\frac{d}{dx}\left(\sin x\right) = \cos x$

$\frac{d}{dx}\left(\cos x\right) = -\sin x$

$\frac{d}{dx}\left(a^x\right) = \ln a \cdot a^x$

$\frac{d}{dx}\left(\tan x\right) = \sec^2 x$

$\frac{d}{dx}\left(\cot x\right) = -\csc^2 x$

$\frac{d}{dx}\left(f(x)\cdot g(x)\right) = f(x)\cdot g'(x) + g(x)\cdot f'(x)$

$\frac{d}{dx}\left(\sec x\right) = \sec x \tan x$

$\frac{d}{dx}\left(\csc x\right) = -\csc x \cot x$

$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)\cdot f'(x) - f(x)\cdot g'(x)}{\left(g(x)\right)^2}$

$\frac{d}{dx}\left(\arcsin x\right) = \frac{1}{\sqrt{1-x^2}}$

$\frac{d}{dx}\left(\arctan x\right) = \frac{1}{1+x^2}$

$\frac{d}{dx}\left(f(g(x))\right) = f'(g(x))\cdot g'(x)$

$\frac{d}{dx}\left(\operatorname{arc\,sec} x\right) = \frac{1}{x\sqrt{x^2-1}}$

Ludwig Wittgenstein