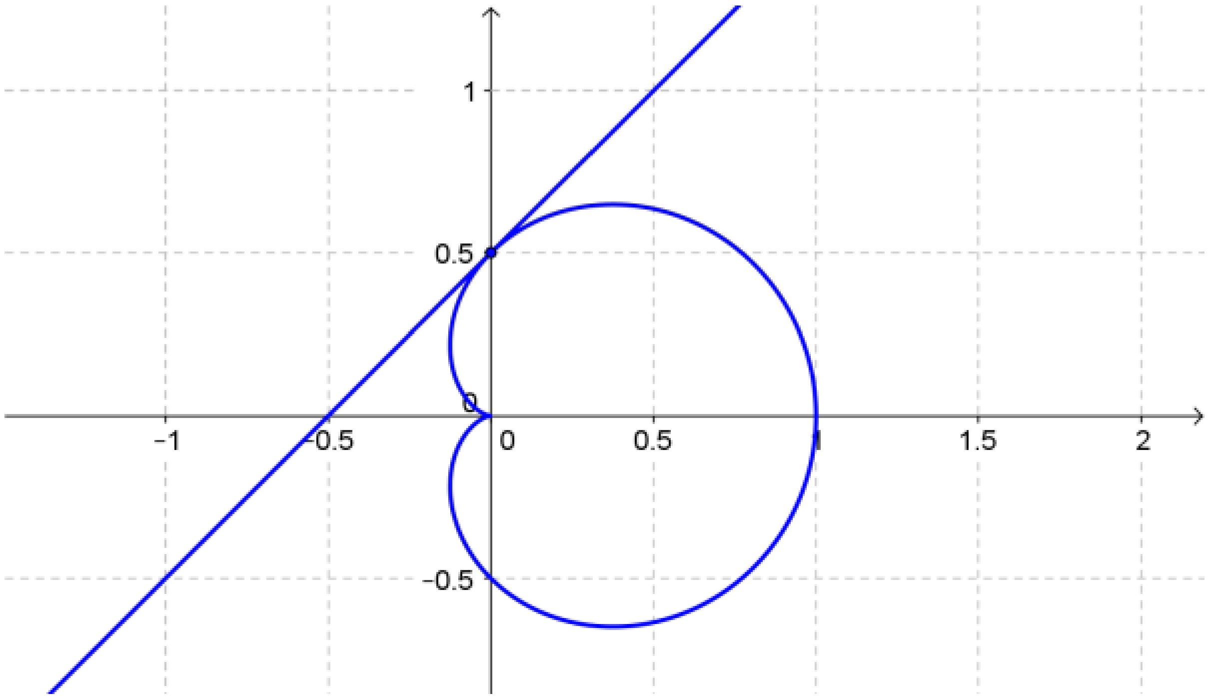
# MATH 161 SOLUTIONS: QUIZ V 25 OCTOBER 2019

**1.** *[Stewart]* Find an equation of the tangent line to the cardioid

$$x^2 + y^2 = (2x^2 + 2y^2 - x)^2$$

at the point P = (0, ½ ).




***Solution:***

We differentiate implicitly the given curve

$$\frac{d}{dx}(x^2 + y^2) = \frac{d}{dx}((2x^2 + 2y^2 - x)^2)$$

So, using the general power rule,

$$2x + 2y\frac{dy}{dx} = 2(2x^2 + 2y^2 - x)\left(4x + 4y\frac{dy}{dx} - 1\right)$$

At the point P = (0, ½ ), we find:

$$0 + 2(\frac{1}{2})\frac{dy}{dx} = 2\left(0 + 2(\frac{1}{2})^2 - 0\right)\left(4(\frac{1}{2})\frac{dy}{dx} - 1\right)$$

Simplifying,

$$\frac{dy}{dx} = 2\left(\frac{1}{2}\right)\left(2\frac{dy}{dx} - 1\right) = 2\frac{dy}{dx} - 1$$

And so, $\frac{dy}{dx} = 1$.

Hence an equation of the tangent line is

$$y - ½ = 1(x - 0)$$

Simplifying, **y = x + ½**

**2.** Using logarithmic differentiation, compute dy/dx for the function

$$y = \frac{x^5}{(1-10x)\sqrt{x^2+2}}$$

***Solution:*** Taking the log of each side,

$$\ln y = ln\left(\frac{x^5}{(1-10x)\sqrt{x^2+2}}\right) = \ln(x^5) - ln\left((1-10x)\sqrt{x^2+2}\right) =$$

$$\ln(x^5) - \left(\ln(1-10x) + \ln\sqrt{x^2+2}\right) =$$
$$5\ln x - \ln(1-10x) - \frac{1}{2}\ln(x^2+2)$$

Next, differentiating each side wrt x, then applying the chain rule,

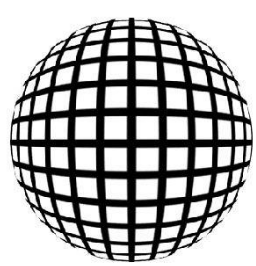
$$\frac{d}{dx}\ln y = \frac{d}{dx}\left(5\ln x - \ln(1-10x) - \frac{1}{2}\ln(x^2+2)\right)$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{5}{x} - \frac{-10}{1-10x} - \frac{1}{2}\frac{2x}{x^2+2} = \frac{5}{x} + \frac{10}{1-10x} - \frac{x}{x^2+2}$$

Hence,

$$\frac{dy}{dx} = \left(\frac{5}{x} + \frac{10}{1-10x} - \frac{x}{x^2+2}\right)y = \left(\frac{5}{x} + \frac{10}{1-10x} - \frac{x}{x^2+2}\right)\left(\frac{x^5}{(1-10x)\sqrt{x^2+2}}\right)$$

**3.** Suppose that a balloon (modeled as a sphere) of radius *r* is being deflated and that, at the moment



when r = 8 cm, its radius is decreasing at the rate of 3 cm/min. How quickly is the *surface area* of the balloon changing at the moment when the balloon's radius is 8 cm?

(*Hint:* The surface area of a sphere of radius *r* is given by $S = 4\pi r^2$.)

***Solution:* *Let*** *r denote the radius of the sphere (in cm).*

***Let*** *S be the surface area of the sphere (in cm$^2$).*

***Given****: dr/dt = - 3 when r = 8.*

***Find****: dS/dt when r = 8*

*Since S = f(r) and r = g(t), we can invoke the Chain Rule:*

$$\frac{dS}{dt} = \frac{dS}{dr}\frac{dr}{dt}$$

*Now:* $\frac{dS}{dr} = 8\pi r$*; evaluating at r = 8:* $\frac{dS}{dr} = 8\pi(8) = 64\pi$

*Since dr/dt = 3, we have:*

$$\frac{dS}{dt} = \frac{dS}{dr}\frac{dr}{dt} = (-3)(64\pi) = \mathbf{-192\,\pi\ cm^2/sec} \approx \mathbf{-603.2\ cm^2/sec}$$

Thus, when r = 8 cm, the surface area is **decreasing at a rate of $603.2\ cm^2/sec$**

## Extra Credit:

*[University of Michigan]* The Kampyle of Eudoxus is a family of curves that was studied by the Greek mathematician and astronomer Eudoxus of Cnidus with the classical problem of doubling the cube. This family of curves is given by

$$a^2x^4 = b^4(x^2 + y^2)$$

where *a* and *b* are nonzero constants and $(x, y) \neq (0,0)$ ---- that is, the origin is not included.

**a)** Find $\frac{dy}{dx}$ for the curve $a^2x^4 = b^4(x^2 + y^2)$

***Solution:*** Using implicit differentiation, we have

$$4a^2x^3 = 2b^4x + 2b^4y\frac{dy}{dx} \quad \text{so} \quad \frac{dy}{dx} = \frac{4a^2x^3 - 2b^4x}{2b^4y}.$$

**b)** Find the coordinates of all points on the curve $a^2x^4 = b^4(x^2 + y^2)$ at which the tangent line is vertical, or show that there are no such points.

***Solution:***

If the tangent is vertical, the slope is undefined.

Setting the denominator from part (a) equal to zero gives y = 0.

Substituting y = 0 in the original equation gives

$$a^2x^4 = b^4x^2$$

And since (0, 0) is excluded, we know that $x \neq 0$ *so* $x^2 = \frac{b^4}{a^2}$. *Hence* $x = \pm\frac{b^2}{a}$.

Thus there are two points on the curve where the tangent line is vertical, viz.

$$\left(\frac{\mathbf{b^2}}{\mathbf{a}}, \mathbf{0}\right) \ \textbf{and} \ \left(-\frac{\mathbf{b^2}}{\mathbf{a}}, \mathbf{0}\right)$$

**c)** Show that when a = 1 and b = 2, there are no points on the curve at which the tangent line is horizontal.

***Solution:*** Using a = 1 and b = 2 in $\frac{dy}{dx}$ from part (a), we have

$$\frac{dy}{dx} = \frac{4x^3 - 32x}{32y}.$$

If the tangent line is horizontal, the slope of the curve is zero. So solving

$$4x^3 - 32x = 4x(x^2 - 8) = 0$$

yields x = 0 or $x = \pm\sqrt{8}$ .

We must check to see if any of these values of *x* correspond to a point on the curve.

Note that when x = 0, y = 0, and this point has been excluded from the family.

If $x = \pm\sqrt{8}$ *, the equation of the curve gives us* $64 = 16(8) + 16y^2$.

So $y^2 = -4$. This equation has no real solution; hence **there are no horizontal tangents** to the given curve.

## DERIVATIVE RULES

$$\frac{d}{dx}\left(x^n\right) = nx^{n-1}$$

$$\frac{d}{dx}\left(a^x\right) = \ln a \cdot a^x$$

$$\frac{d}{dx}\left(f(x)\cdot g(x)\right) = f(x)\cdot g'(x) + g(x)\cdot f'(x)$$

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)\cdot f'(x) - f(x)\cdot g'(x)}{\left(g(x)\right)^2}$$

$$\frac{d}{dx}\left(f(g(x))\right) = f'(g(x))\cdot g'(x)$$

$$\frac{d}{dx}\left(\ln x\right) = \frac{1}{x}$$

$$\frac{d}{dx}\left(\sin x\right) = \cos x \qquad \frac{d}{dx}\left(\cos x\right) = -\sin x$$

$$\frac{d}{dx}\left(\tan x\right) = \sec^2 x \qquad \frac{d}{dx}\left(\cot x\right) = -\csc^2 x$$

$$\frac{d}{dx}\left(\sec x\right) = \sec x \tan x \qquad \frac{d}{dx}\left(\csc x\right) = -\csc x \cot x$$

$$\frac{d}{dx}\left(\arcsin x\right) = \frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}\left(\arctan x\right) = \frac{1}{1+x^2}$$

$$\frac{d}{dx}\left(\operatorname{arc}\sec x\right) = \frac{1}{x\sqrt{x^2-1}}$$

$$\frac{d}{dx}\left(\sinh x\right) = \cosh x \qquad \frac{d}{dx}\left(\cosh x\right) = \sinh x$$