Instructions: Answer any 4 of the following 5 problems. You may answer all 5 to earn extra credit.

1. *[8 pts]* (Stewart) The sum of two positive real numbers is 16. What is the smallest possible value of the sum of their squares? *(Give complete solution.)*

Solution:

Let x and y be two positive real numbers.

Assume that x + y = 16.

Let F(x, y) = $x^2 + y^2$. Our goal is to **minimize F.**

To express F as a function of a single variable, replace y by 16 – x,

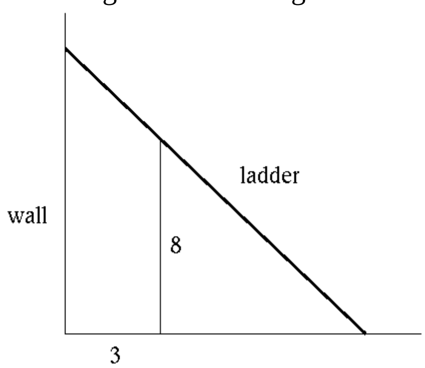
Hence F(x) = $F(x) = x^2 + y^2 = x^2 + (16 - x)^2$

The constraints upon x are $0 \leq x \leq 16$.

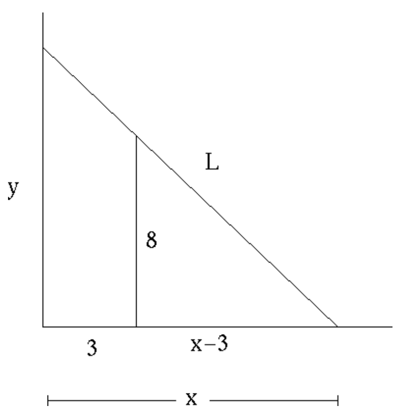
Now F(x) is a concave up parabola with vertex at (8, 128).

Since x =8 satisifes the constrains place upon x, we have found our global minimum at x = 8, at which F(8) = **128**.

2. *[8 pts]* (UC Davis) Odette wishes to find the length of the shortest ladder that will reach over an 8-foot high fence to a large wall that is 3 feet behind the fence. Introduce variables. State your goal.



Answer: Let x and y be the lengths (in feet) as indicated in the following diagram.



Our goal is to express L as a function of x and then *minimize L*.

Now, L $is\ a\ function\ of\ x\ and\ y$: $L(x,y) = \sqrt{x^2+y^2}$

We will eliminate the variable y by using similar triangles:

$$\frac{y}{x} = \frac{8}{x-3}$$

so that

$$y = \frac{8x}{x-3}.$$

$$L(x) = \sqrt{x^2+y^2} = \sqrt{x^2 + \left(\frac{8x}{x-3}\right)^2}$$

The constraints on x are: $8 < x < \infty$

FYI: The final answers are: $x \approx 8.77\ ft, y \approx 16.67\ ft, and\ L \approx 17.64\ ft.$

3. *[8 pts]* Harvey Swick is planning to build a rectangular garden in which to grow pumpkins. Two opposite sides of the garden will be bordered by lilacs, which cost $70 per meter. The other pair of opposite sides will be bounded by a wooden fence that costs $100 per meter. The area enclosed by his garden must be 400 square meters. Harvey wishes to minimize his expenditures on shrubs and fencing. Your goal is to find the dimensions of the garden that *minimizes his total cost*? (Be certain to identify your variables, draw a diagram, and use appropriate units.)
Define your variables. Find a function of one variable that needs to be minimized. Write the constraints on your variable. Stop at this point.

Solution: Let x be the length (in meters) of (one of the two) sides that is planted with lilacs, and let y be the length (in meters) of (one of the two) sides to be fenced.

Then the total cost is $C(x, y) = 2(70x + 100y)$ dollars.

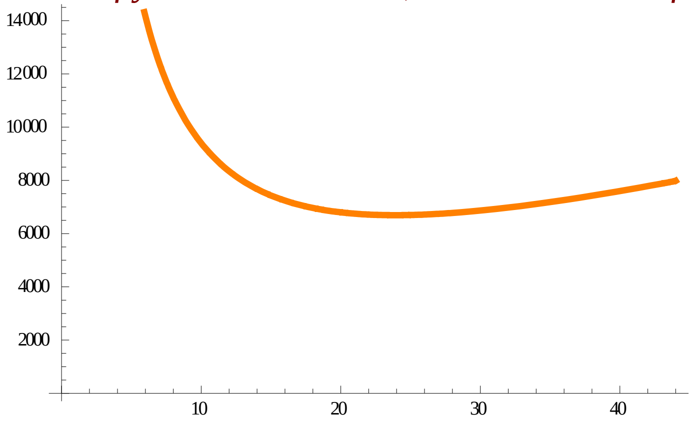
The area is fixed, so we have $xy = 400$ square meters.

Our goal is to minimize C. To express C as a function of x alone, we replace y by 400/x in the equation for C to obtain:

$$C(x) = 2(70x + 40000/x) = 20(7x + 4000/x).$$

Clearly, $0 < x < \infty$ is the domain (or constraints) of C.

FYI: If you are interested, here is the rest of the solution:



To determine local/global extrema, we compute:

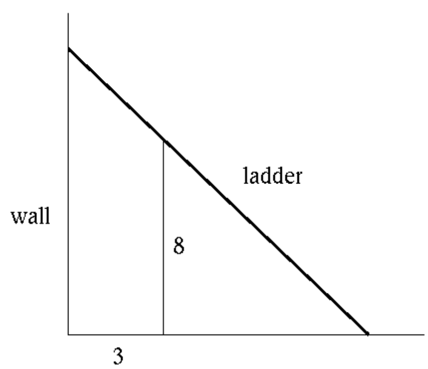
$$dC/dx = 20(7 - 4000/x^2)$$

Factoring:

$$\frac{dC}{dx} = \frac{20}{x^2}(7x^2 - 4000) = \frac{20}{x^2}\left(x\sqrt{7} - 20\sqrt{10}\right)\left(x\sqrt{7} + 20\sqrt{10}\right)$$

Hence C is increasing on $(0,\ 20\sqrt{10/7})$ and decreasing on $(20\sqrt{10/7},\ \infty)$. Thus, applying the first derivative test, a global minimum is achieved when x = $20\sqrt{10/7} \approx 23.9$ meters. For the optimal x-value, we find that $y = 400/x \approx 400/\ 23.9 = 16.7$ meters.

4. *[8 pts]* Odette wishes to find the length of the shortest ladder that will reach over an 8-foot high fence to a large wall that is 3 feet behind the fence.



Solution: Let *R* be the radius (in feet) of the tank and let *H* be the height (in feet) of the tank.

Then the volume of the tank equals $\pi R^2 H$, which must equal 50.

The total surface area is given by:

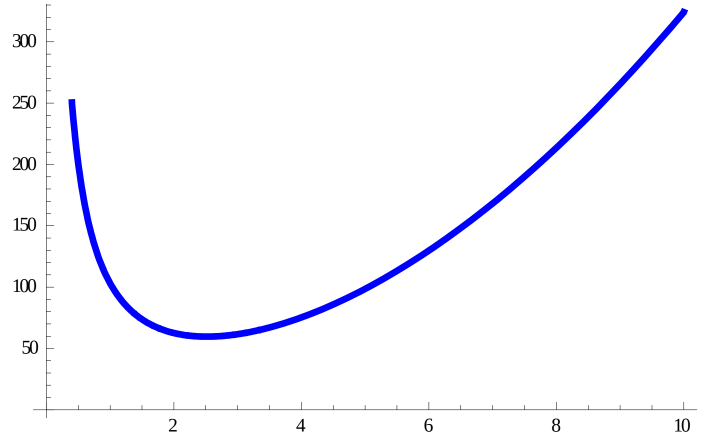
$$S = 2\pi RH + \pi R^2.$$

Our goal is to minimize S. To express S as a function of a single variable, we replace H by $50/(\pi R^2)$, to obtain:

$$S = S(R) = 2\pi R(50/(\pi R^2)) + \pi R^2 = 100/R + \pi R^2$$

The constraints on R: $0 < R < \infty$.

FYI: Here is the remaining portion of a complete solution.

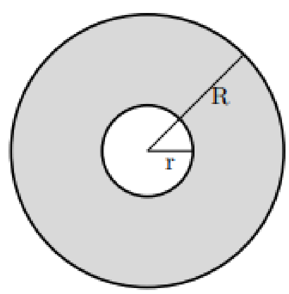


To determine local/global extrema, we compute:

$$\frac{dS}{dR} = 2\pi R - \frac{100}{R^2} = \frac{2\pi R^3 - 100}{R^2} = \frac{1}{2\pi R^2}\left(R^3 - \frac{50}{\pi}\right)$$

The unique critical point is R = $\sqrt[3]{50/\pi} \approx 2.52$. Clearly, if 0 < R < 2.52, then S is decreasing and when R > 2.52, S is increasing. Hence, using the first derivative test, S achieves a global minimum at R = 2.52 feet. Now, when R = **2.52** feet, H = 50/{π (2.52)^2} = **2.52** feet.

5. *[8 pts] (University of Michigan)* Albertine is machining a metal washer to fix her broken-down motorcycle. A washer is a flat, circular piece of metal with a hole in the middle. Albertine's washer is represented by the shaded region in the figure below. The washer has an inner radius of *r* centimeters and an outer radius of *R* centimeters. The area of the washer must be exactly 5 square centimeters, and *r* must be at least 1 centimeter.



a. *[3 points]* Find a formula for *r* in terms of *R*.

Solution: The area of the washer is the difference between the outer circle's area and inner circle's area. So, since this must be 5 square centimeters we have $\pi R^2 - \pi r^2 = 5$, so $r^2 = \frac{\pi R^2 - 5}{\pi}$, and $r = \sqrt{\frac{\pi R^2 - 5}{\pi}}$.

b. *[2 points]* The structural integrity of the washer depends on both its inner radius and its outer radius.
Specifically, the structural integrity is given by the equation

$$S = 32R(\ln(rR + 1) + 7)$$

Express *S* as a function of *R*. *Your answer should not include r.*

Solution: We substitute our answer from part a. into the formula for S.

Answer: $S(R) =$ $32R\left(\ln\left(R\sqrt{\frac{\pi R^2-5}{\pi}}+1\right)+7\right)$

c. *[3 points]* What are the constraints of *R* in the context of this problem? You may give your answer as an interval or using inequalities.

Solution: We are told that r must be at least 1. When $r = 1$, we have

$$\pi R^2 - \pi r^2 = 5$$
$$\pi R^2 - \pi = 5$$
$$R^2 = \frac{5+\pi}{\pi}$$
$$R = \sqrt{\frac{5+\pi}{\pi}}.$$

This is the smallest possible value of R, because if we make r larger, R must also be made larger so that the area of the washer can remain 5 square centimeters. There is no upper bound on how large R can be.

Answer: $\left[\sqrt{\frac{5+\pi}{\pi}}, \infty\right)$