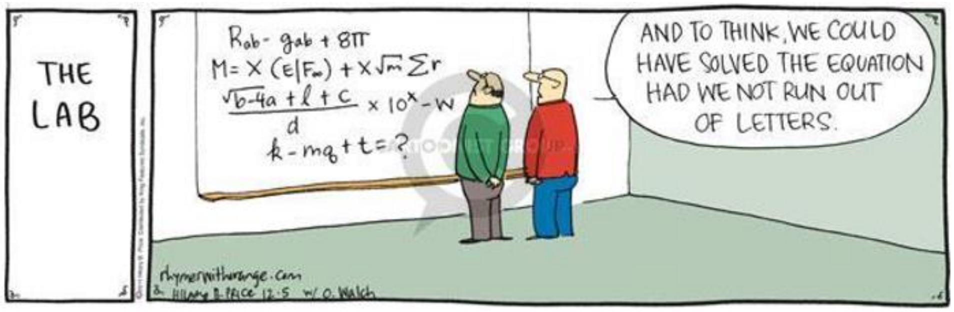
MATH 161 REVISED SOLUTIONS: TEST I

27 SEPTEMBER 2019



I'm very good at integral and differential calculus,
I know the scientific names of beings animalculous;
In short, in matters vegetable, animal, and mineral,
I am the very model of a modern Major-General.
About binomial theorems, I'm teeming with a lot of news,
With many cheerful facts about the square on the hypotenuse.

- W. S. Gilbert, **The Pirates of Penzance** (1879)

Instructions: *Answer any 7 of the following 9 questions. You may solve more than 7 to obtain extra credit.*

1. Using *the limit definition of the derivative*, write an explicit expression for the *derivative* of the function

$$L(x) = (1+x^2)^{3x-4} \ \ at\ x = 2.$$

Do not try to calculate this derivative.

Solution: $\frac{dy}{dx} = \lim_{h\to 0} \frac{L(2+h)-L(2)}{h} = \lim_{h\to 0} \frac{\left(1+(2+h)^2\right)^{3(2+h)-4} - (1+4)^2}{h} =$

$\lim_{h\to 0} \frac{\left(1+(2+h)^2\right)^{3(2+h)-4} - 25}{h}$

2. *Using only the definition of derivative* compute $f'(7)\ \ if\ f(x) = \sqrt{x+2}$. (Use no short-cuts.)

Solution: $\lim_{h\to 0} \frac{f(7+h)-f(7)}{h} = \lim_{h\to 0} \frac{\sqrt{9+h}-\sqrt{9}}{h} = \lim_{h\to 0} \frac{\sqrt{9+h}-3}{h}$

$$\frac{\sqrt{9+h}-3}{h} = \left(\frac{\sqrt{9+h}-3}{h}\right)\left(\frac{\sqrt{9+h}+3}{\sqrt{9+h}+3}\right) = \frac{9+h-9}{h\left(\sqrt{9+h}+3\right)} =$$

$$\frac{h}{h\left(\sqrt{9+h}+3\right)}=\frac{1}{1\left(\sqrt{9+h}+3\right)}\to\frac{1}{6}$$

3. The annual number of respiratory infections in a city is a function of the amount of carbon in the atmosphere above that city.
Let R(p) be the annual number of respiratory infections in Alphaville when there are *p* thousand tons of carbon in the atmosphere above the city.
Let C(k) be the healthcare cost, in thousands of dollars, of treating *k* respiratory infections.
The functions R(p) and C(k) are differentiable and invertible.

a) Give a practical interpretation of the equation $R^{-1}(212) = 24$.

Solution: There are 212 respiratory infections annually in Alphaville when there are 24 thousand tons of carbon in the atmosphere above the city.

b) Give a *practical interpretation* of the equation C(R(17)) = 650. (Do not use any technical terms.)

Solution: When there are 17 thousand tons of carbon in the atmosphere above Alphaville, the resulting annual healthcare cost for respiratory infections in Alphaville is 650 thousand dollars.

c) Write a mathematical equation that represents the following statement:

The healthcare cost of treating 165 respiratory infections is 400 thousand dollars more than the healthcare cost of treating 130 respiratory infections.

Solution: C(165) = 400 + C(130)

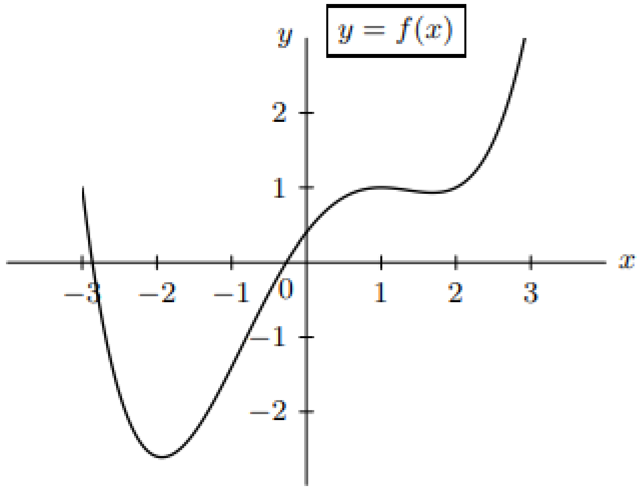
d) Complete the following sentence using the fact that $R'(38) = 4$.

If the amount of carbon in the atmosphere above Alphaville is reduced from 41 thousand tons to 38 thousand tons, . . .

Solution:

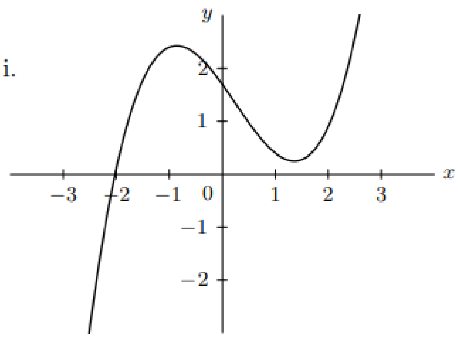
…then the annual number of respiratory infections in Alphaville will decrease by approximately 12.

4. a) Below is the graph of a function y = f(x).



There are six graphs shown below. *Circle the one graph that could be the graph of the derivative f ' (x).*

i.



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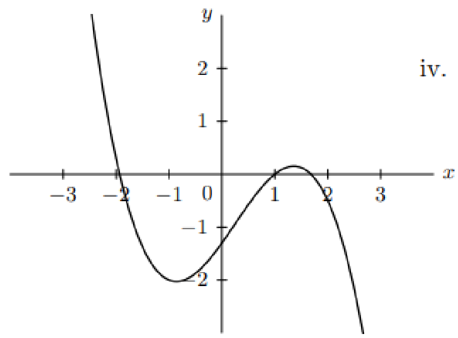

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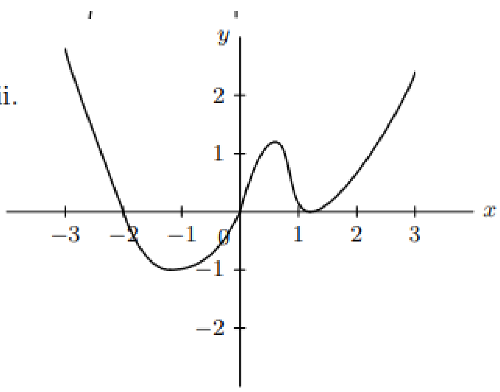

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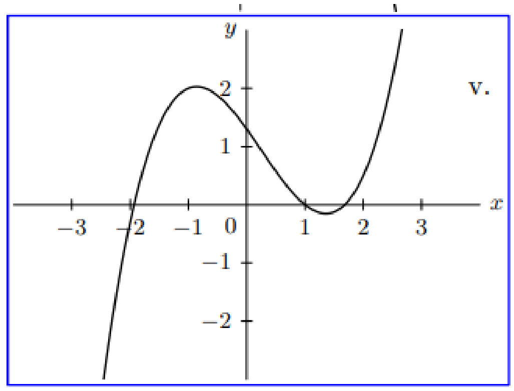
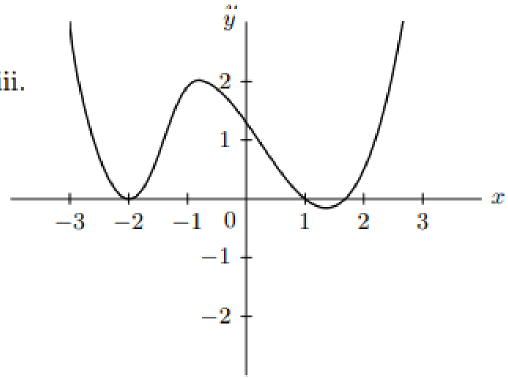
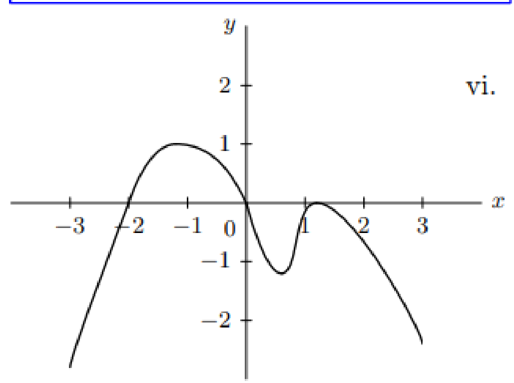


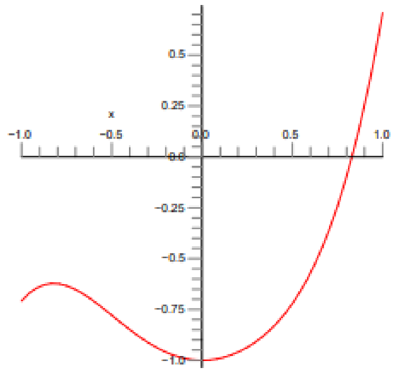
iii.



vi.

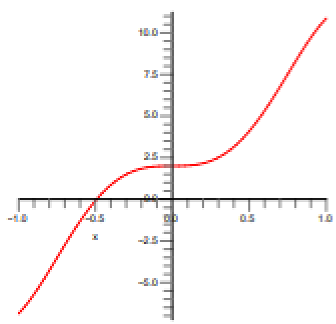


b) The graph of y = g(x) is given by the figure below.

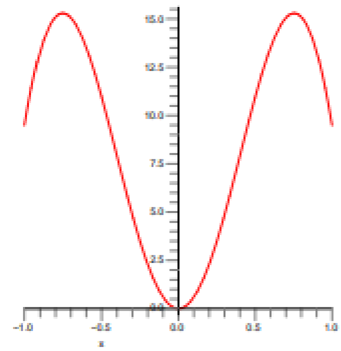


The graphs of the following functions include the graph of g ′ and three additional graphs. On the lines beneath the appropriate graphs, clearly identify the graph of *g ′*. (No explanation necessary.) *For each of the other graphs, give any feature of the graph (e.g., its behavior on an interval or at a point) which disqualifies that graph as a candidate for the derivative of g. (No need to list all reasons!)*

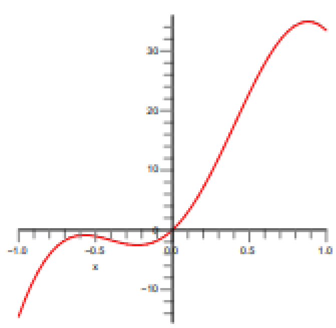
Solution:



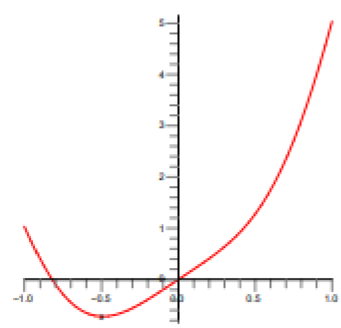
This is g''



Not g'—On (-0.85,0) g is decreasing, so g' must be negative. This graph is positive on (-0.85,0).

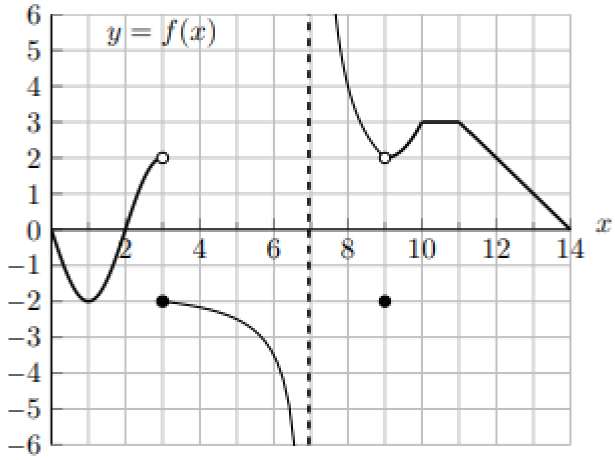


Not g'—On (-1,-0.85) g is increasing, so g' must be positive. This graph is negative on (-1,-0.85).



This is g'

5. Below is a portion of the graph of an ***odd*** function y = f(x), and the formula for a function g(x). Note that f(x) is linear for 11 < x < 14. If the limit does not exist, write DNE.



$$g(x) = \frac{x^4 + 1}{e^{x^2}}$$

(a) Find g(f(2))

Answer: **1**

(b) $\lim\limits_{h\to 0} \frac{f(12+h)-2}{h}$

Answer: **-1**

Note: This is the slope of the line joining the two points (12, 2) and (12, f(12+h)).

(c) $\lim\limits_{x\to -1}(f(x) + g(x))$

Answer:

$$\mathbf{2 + \frac{2}{e}}$$

$(d)\ \lim\limits_{x\to 7}\ f(x)$

Answer: **DNE**

$(e)\ \lim\limits_{x\to \infty}\ g(x)$

Answer: 0

$(\boldsymbol{f})\ \lim\limits_{x\to -9}\ f(x)$ Note f(-x) = -f(x) since f is an *odd* function.

Answer: **-2**

(g) Extra credit: $\lim_{x \to 3} g(f(x))$

Answer: $\mathbf{\frac{17}{e^4}}$

(h) Extra credit: $\lim_{x \to 11+} f(f(x))$

Answer: **-2**

6. In the following table, both *f* and *g* are differentiable functions of *x*. In addition, g(x) is an invertible function.

x	2	3	4	5
$f(x)$	7	6	2	9
$f'(x)$	-2	1	3	2
$g(x)$	1	4	7	11
$g'(x)$	1	2	3	2

a) $If\ p(x) = 3f(x) - 9g(x) - x^3 + e^9\ find\ p'(2).$

Solution: $p'(x) = 3f'(x) - 9g'(x) - 3x^2$ (Notice that e^9 is a constant!)

So $p'^{(2)} = 3f'^{(2)} - 9g'(2) - 3\ 2^2 = 3(-2) - 9(1) - 12 = \mathbf{-27} =$

b) $If\ h(x) = \frac{g(x)}{f(x)}\ find\ h'(4).$

Solution: $h'(x) = \frac{g(x)f'(x) - f(x)g\prime(x)}{(f(x))^2}$

So : $h'^{(4)} = \frac{f(4)g\prime(4) - g(4)f'(4)}{(f(4))^2} = \frac{2(3) - 7(3)}{2^2} = -\mathbf{\frac{15}{4}}$

c) $If\ k(x) = f(x)g(x)\ find\ k'(2).$

Solution: $k'(x) = f(x)g'(x) + f'(x)g(x)$

And so $k'(2) = f(2)g'(2) + f'(2)g(2) = \quad 7(1) + (-2)(1) = \mathbf{5}$

7. Let f(x) be a differentiable function defined for all real numbers and assume that

$$f\ '(x) = \frac{(2x-3)(x-2)^2}{(x+5)^{\frac{1}{3}}}.$$

(a) Find the x-coordinate(s) of any and all critical point(s) of f(x).

Solution: Setting

$$f\ '(x) = 0, we\ have\ x = \frac{3}{2}\ and\ x = 2\ are\ critical\ points.$$

Note that even though the derivative of *f* does not exist at x = -5, it doesn't follow that *f* is undefined at x = -5. So we should treat x = -5 as a critical point as well.

(b) On which interval(s) is *f* increasing? On which interval(s) is *f* decreasing?

Solution: $f\ '^{(x)} changes\ sign\ at\ x = \frac{3}{2}\ and\ at\ x = \ -5.$

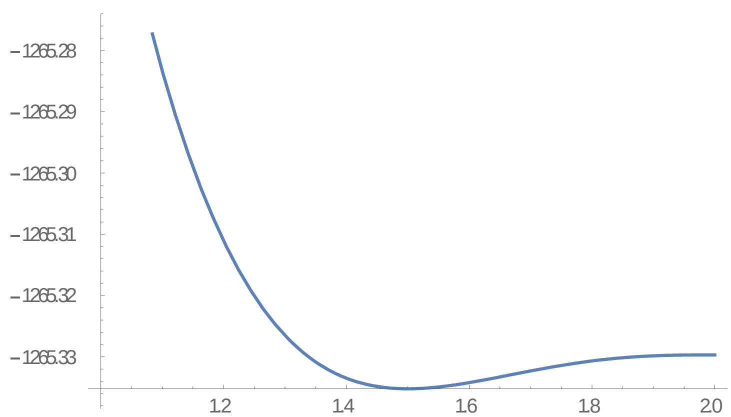
$f'\ is\ positive\ on\ \left(\frac{3}{2}, \infty\right), and\ on\ (-\infty, -5), and\ negative\ on\ (-5, \frac{3}{2})$

Hence f is increasing on

$$(-\infty, -5)\ and\ \left(\frac{3}{2}, \infty\right).\ f\ is\ decreasing\ on\ \left(-5,\ \frac{3}{2}\right).$$

(c) Classify each critical point of *f* as a local max, a local min, or neither.

Solution: Using the first derivative test, we find that g has a **local maximum** at x = -5. Also a **local min** at x = $\frac{3}{2}$.



(d) Find the *slope* of the tangent line to y = f(x) at x = 3.

Solution:

$$f'(x) = \frac{(2x-3)(x-2)^2}{(x+5)^{\frac{1}{3}}}$$

So

$$f'(3) = \frac{(2(3)-3)(3-2)^2}{(3+5)^{\frac{1}{3}}} = \mathbf{\frac{3}{2}}.$$

(e) Find the *slope* of the *normal line* to y = f(x) at x = 3.

Solution:

The slope of the normal line is the negative reciprocal of 3/2. Hence $-\mathbf{\frac{2}{3}}$

8. Compute each of the following limits. *Justify your reasoning.*

(a) $\lim_{x\to\infty} \frac{\sin x + 4\cos x + 1}{x^2}$

Solution: This limit is 0 since the denominator increases without bound, yet the numerator is trapped between -4 and 6.

(b) $\lim_{x\to\infty} \frac{(2x+9)(3x^2+99)^3}{(x+7)^4(2x-77)^3}$

Solution: Using highest order of magnitude method,

$$\frac{(2x+9)(3x^2+99)^3}{(x+7)^4(2x-77)^3} \cong \frac{(2x)(3x^2)^3}{(x)^4(2x)^3} = \frac{54x^7}{8x^7} = \mathbf{\frac{27}{4}}$$

(c) $\lim_{x\to 0} \frac{\sin 7x}{\tan 3x}$

Solution:

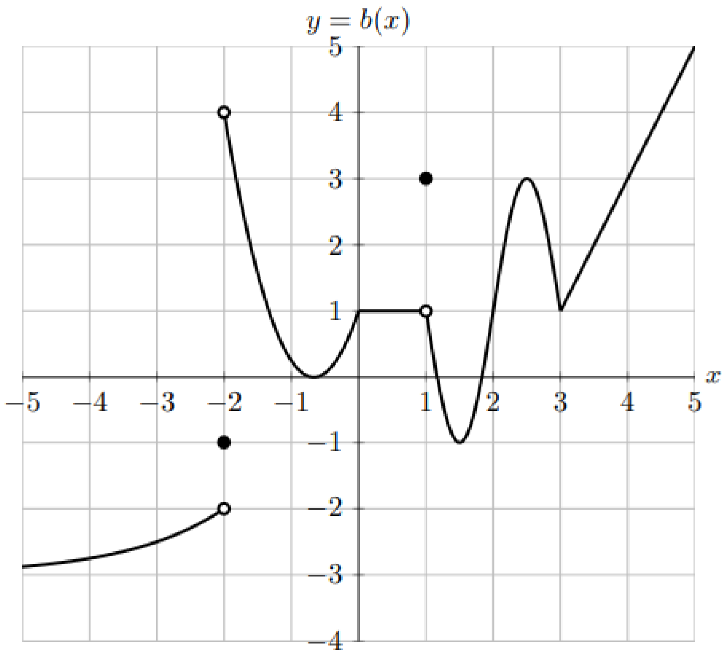
$$\frac{\sin 7x}{\tan 3x} = \frac{\sin 7x}{\sin 3x}\cos 3x \to \frac{7}{3}\ (1) = \frac{7}{3}.$$

(d) $\lim_{x\to 2} \frac{\frac{1}{4}-\frac{1}{x^2}}{x-2}$

Solution:

$$\frac{\frac{1}{4}-\frac{1}{x^2}}{x-2} = \frac{\frac{x^2-4}{4x^2}}{x-2} = \frac{(x+2)(x-2)}{4x^2(x-2)} = \frac{x+2}{4x^2} \to \frac{4}{16} = \frac{1}{4} \quad as \;\; x \to 2$$

9. A portion of the function b(x) is depicted in the graph below. This function is defined for all real numbers *x*.



Find the exact value of the limits below. If any of the limits does not exist, write “DNE.” If there is not enough information provided to you to answer the question, write “NI.” You do not need to show your work.

Answers:

a) $\lim_{x\to -2} b(x)\;\; DNE$

b) $\lim_{x\to -2^-} b(x) =$ -2

c) $\lim_{x\to 1} b(x) \;\; = 1$

d) $\lim_{m\to 0} \frac{b(4+m)-b(4)}{m} \; = \; 2$

This is the *slope* of the line segment joining the points (3, 1) and (5, 5)

e) $\lim_{s\to 0-} b(b(s)) = 1$

f) $\lim_{x\to 0+} b(b(x)) = 3$

g) $\lim_{x\to\infty} b\left(-2+\frac{1}{x}\right) = 4$ since $\frac{1}{x} \to 0+ \ as\ x \to \infty$.

DERIVATIVE RULES

$$\frac{d}{dx}\left(x^n\right)=nx^{n-1}$$

$$\frac{d}{dx}\left(\sin x\right)=\cos x$$

$$\frac{d}{dx}\left(\cos x\right)=-\sin x$$

$$\frac{d}{dx}\left(a^x\right)=\ln a\cdot a^x$$

$$\frac{d}{dx}\left(\tan x\right)=\sec^2 x$$

$$\frac{d}{dx}\left(\cot x\right)=-\csc^2 x$$

$$\frac{d}{dx}\left(f(x)\cdot g(x)\right)=f(x)\cdot g'(x)+g(x)\cdot f'(x)$$

$$\frac{d}{dx}\left(\sec x\right)=\sec x\tan x$$

$$\frac{d}{dx}\left(\csc x\right)=-\csc x\cot x$$

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right)=\frac{g(x)\cdot f'(x)-f(x)\cdot g'(x)}{\left(g(x)\right)^2}$$

$$\frac{d}{dx}\left(\arcsin x\right)=\frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}\left(\arctan x\right)=\frac{1}{1+x^2}$$

$$\frac{d}{dx}\left(f(g(x))\right)=f'(g(x))\cdot g'(x)$$

$$\frac{d}{dx}\left(\operatorname{arc\,sec} x\right)=\frac{1}{x\sqrt{x^2-1}}$$

$$\frac{d}{dx}\ln \mathrm{x}=\frac{1}{\mathrm{x}}$$