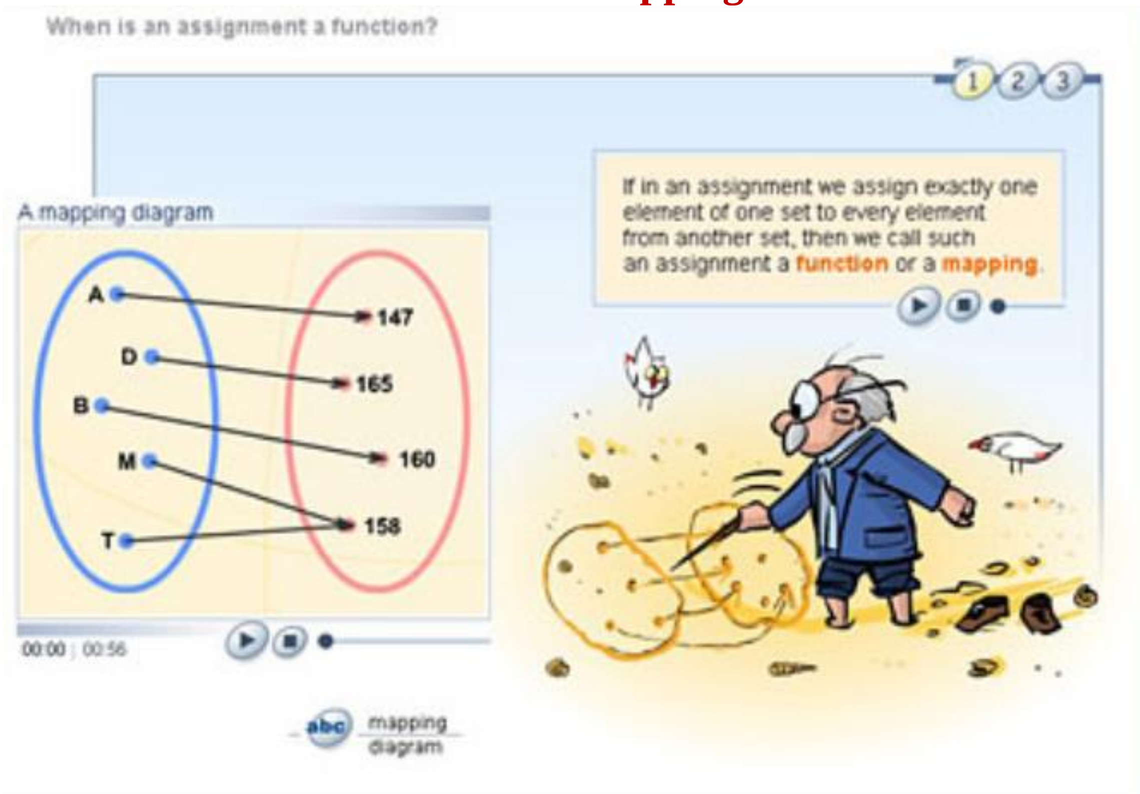
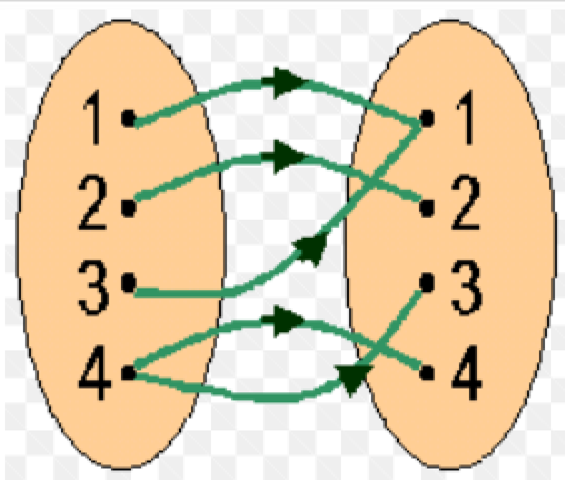
Math 201 Class Discussion: Mappings 19 November 2019





I Let X and Y be non-empty sets, and let F: X→Y be a function (or mapping). What does it mean to say that *F* is well-defined? an injection? a surjection? a bijection?

II Let a b and denote real numbers. Which (if any) of the following functions are ***well-defined***?

a) $f(a+b)=b$

b) $f(a+b)=\sin(a+b)$

c) $f(a+b)=a+2b$

d) $f(a+b)=ab$

III In the following, let **N**, **Z**, **Q**, **R** denote the set of natural numbers, the set of integers, the set of rational numbers, and the set of real numbers, respectively. For each mapping below, first determine if it is well-defined. If so, then determine if it enjoys any of the properties of being injective, surjective, bijective.

(a) F: **N**→**N** given by F(n) = n + 1

(b) G: **N**→**N** given by G(n) = n – 1

(c) H: **Z**→**N** given by H(m) = |m|

(d) f: **Z**→**N** given by f(m) = |m| + 1

(e) α: **Q**→**Z** given by $\alpha\left(\frac{a}{b}\right) = a + b$

(f) β: **Q**→**Z** given by β(x) = ab where x = a/b (where *a* and *b* are nonnegative) or –a/b (where *a* and *b* are nonnegative) and gcd(a,b)=1

(g) F: **N**→**N** given by $F(n) = n^2$

(h) G: **R**→**R** given by G(x) = (x – 1)(x – 2)(x – 3)

(i) H: **Z**→**Z** given by H(m) = z + 11

(j) id: **X**→**X** given by id(m) = m

(k) p: **N**→**Q** given by p(j) = 1/j

(l) F: **N**→**Q** given by F(n) = n/13

(m) z: **N**→**N** given by z(m) = sum of the digits in the decimal representation of *m*.

IV

1. Given the following three pairs of sets

- $\{1,2,3\}$ and $\{4,5,6\}$
- $\mathbb{N}$ and $\mathbb{N}$
- $\mathbb{R}$ and $\mathbb{R}$

For each of the above pairs of functions find four functions f_1, f_2, f_3, f_4 from the first set to the second set such that

(a) f_1 is neither injective nor surjective.

(b) f_2 is injective but not surjective.

(c) f_3 is surjective but not injective

(d) f_4 is bijective

2. Find examples of the following functions f.

a) f maps $\mathbb{R}$ to $\{1,2,3\}$

b) f maps $\mathbb{N}$ to $\mathbb{R}$

c) f maps $\mathbb{R}\times\mathbb{R}$ to $\mathbb{R}$

d) f maps $\mathbb{R}$ to $\mathbb{R}\times\mathbb{R}$

e) f maps $\{a,b,c\}$ to $[0,1]$

3. **(Injections, Surjections, and Bijections)** Which of the following functions are intective, surjective, and bijective on their respective domains. Take the domains of the functions as those values of x for which the function is well-defined.

a) $f(x)=x^3-2x+1$

b) $f(x)=\dfrac{x+1}{x-1}$

c) $f(x)=\begin{cases} x^2 & x\le 0 \\ x+1 & x>0 \end{cases}$

d) $f(x)=\dfrac{1}{x^2+1}$

V Let X, Y, Z be non-empty sets. Assume that F: X→Y and G: Y→Z are (well-defined) mappings. For each of the following statements give a proof or counterexample.

(a) If F and G are injective then G∘F is injective.

(b) If F and G are surjective then G∘F is surjective.

(c) If F and G are bijective then G∘F is bijective.

(d) If G∘F is injective, then G is injective.

(e) If G∘F is injective, then F is injective.

(f) If G∘F is surjective, then F is surjective.

(g) If G∘F is surjective, then G is surjective.

VI Let f: X→Y be a function. When does f possess an inverse?

For each of the following, decide if an iverse exists. If yes, find it.

(a) f: N→Z defined by f(j) = -j

(b) f: R→ R defined by $f(x) = x^5$

(c) g: R→ (0, ∞) defined by $g(x) = e^x$

(d) h: Z → Z defined by h(j) = j + 13

(e) f: $(0, \infty) \rightarrow (0, \infty)$ defined by $f(x) = 1/x$

(f) G:$[0, \infty) \rightarrow [0, \infty)$ defined by $f(x) = x^2$

VII Find a bijection from

(a) R to R

(b) N to Z

(c) Z to N

(d) [0, 5] to [7, 17]

(e) (0, 1) to R

(f) N to Q∩ (0, 1)

VIII Show that a mapping G: X→ Y may also be regarded as a mapping from P(X) to P(Y).

University of Maine notes with many examples.

Exercises for Section 12.5

1. Check that the function $f : \mathbb{Z} \to \mathbb{Z}$ defined by $f(n) = 6 - n$ is bijective. Then compute f^{-1}.
2. In Exercise 9 of Section 12.2 you proved that $f : \mathbb{R} - \{2\} \to \mathbb{R} - \{5\}$ defined by $f(x) = \frac{5x+1}{x-2}$ is bijective. Now find its inverse.
3. Let $B = \{2^n : n \in \mathbb{Z}\} = \{\ldots, \frac{1}{4}, \frac{1}{2}, 1, 2, 4, 8, \ldots\}$. Show that the function $f : \mathbb{Z} \to B$ defined as $f(n) = 2^n$ is bijective. Then find f^{-1}.
4. The function $f : \mathbb{R} \to (0, \infty)$ defined as $f(x) = e^{x^3+1}$ is bijective. Find its inverse.
5. The function $f : \mathbb{R} \to \mathbb{R}$ defined as $f(x) = \pi x - e$ is bijective. Find its inverse.
6. The function $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}$ defined by the formula $f(m, n) = (5m + 4n, 4m + 3n)$ is bijective. Find its inverse.
7. Show that the function $f : \mathbb{R}^2 \to \mathbb{R}^2$ defined by the formula $f(x, y) = ((x^2 + 1)y, x^3)$ is bijective. Then find its inverse.
8. Is the function $\theta : \mathscr{P}(\mathbb{Z}) \to \mathscr{P}(\mathbb{Z})$ defined as $\theta(X) = \overline{X}$ bijective? If so, what is its inverse?
9. Consider the function $f : \mathbb{R} \times \mathbb{N} \to \mathbb{N} \times \mathbb{R}$ defined as $f(x, y) = (y, 3xy)$. Check that this is bijective; find its inverse.
10. Consider $f : \mathbb{N} \to \mathbb{Z}$ defined as $f(n) = \frac{(-1)^n(2n-1)+1}{4}$. This function is bijective by Exercise 18 in Section 12.2. Find its inverse.

Exercises for Section 12.6

1. Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined as $f(x) = x^2 + 3$. Find $f([-3,5])$ and $f^{-1}([12,19])$.
2. Consider the function $f : \{1,2,3,4,5,6,7\} \to \{0,1,2,3,4,5,6,7,8,9\}$ given as

$$f = \{(1,3),(2,8),(3,3),(4,1),(5,2),(6,4),(7,6)\}.$$

   Find: $f(\{1,2,3\})$, $f(\{4,5,6,7\})$, $f(\emptyset)$, $f^{-1}(\{0,5,9\})$ and $f^{-1}(\{0,3,5,9\})$.
3. This problem concerns functions $f : \{1,2,3,4,5,6,7\} \to \{0,1,2,3,4\}$. How many such functions have the property that $|f^{-1}(\{3\})| = 3$?
4. This problem concerns functions $f : \{1,2,3,4,5,6,7,8\} \to \{0,1,2,3,4,5,6\}$. How many such functions have the property that $|f^{-1}(\{2\})| = 4$?
5. Consider a function $f : A \to B$ and a subset $X \subseteq A$. We observed in Section 12.6 that $f^{-1}(f(X)) \neq X$ in general. However $X \subseteq f^{-1}(f(X))$ is always true. Prove this.
6. Given a function $f : A \to B$ and a subset $Y \subseteq B$, is $f(f^{-1}(Y)) = Y$ always true? Prove or give a counterexample.
7. Given a function $f : A \to B$ and subsets $W, X \subseteq A$, prove $f(W \cap X) \subseteq f(W) \cap f(X)$.
8. Given a function $f : A \to B$ and subsets $W, X \subseteq A$, then $f(W \cap X) = f(W) \cap f(X)$ is *false* in general. Produce a counterexample.
9. Given a function $f : A \to B$ and subsets $W, X \subseteq A$, prove $f(W \cup X) = f(W) \cup f(X)$.
10. Given $f : A \to B$ and subsets $Y, Z \subseteq B$, prove $f^{-1}(Y \cap Z) = f^{-1}(Y) \cap f^{-1}(Z)$.
11. Given $f : A \to B$ and subsets $Y, Z \subseteq B$, prove $f^{-1}(Y \cup Z) = f^{-1}(Y) \cup f^{-1}(Z)$.
12. Consider $f : A \to B$. Prove that f is injective if and only if $X = f^{-1}(f(X))$ for all $X \subseteq A$. Prove that f is surjective if and only if $f(f^{-1}(Y)) = Y$ for all $Y \subseteq B$.
13. Let $f : A \to B$ be a function, and $X \subseteq A$. Prove or disprove: $f(f^{-1}(f(X))) = f(X)$.
14. Let$f : A \to B$ be a function, and $Y \subseteq B$. Prove or disprove: $f^{-1}(f(f^{-1}(Y))) = f^{-1}(Y)$.