**Math 201 Class Discussion: Mappings, continued**

**21 November 2019**

**I (Injections, Surjections, and Bijections)** Which of the following functions are injective, surjective, and bijective on their respective domains. Take the domains of the functions as those whose values of x for which the function is well-defined.



 **II** Let X, Y, Z be non-empty sets. Assume that F: X→Y and G: Y→Z  are (well-defined) mappings. For each of the following statements give a proof or counterexample.

1. If F and G are injective, then G$∘$F is injective.
2. If F and G are surjective, then G$∘$F is surjective.
3. If F and G are bijective, then G$∘$F is bijective.
4. If G$∘$F is injective, then G is injective.
5. If G$∘$F is injective, then F is injective.
6. If G$∘$F is surjective, then F is surjective.
7. If G$∘$F is surjective, then G is surjective.

**III** Let f: X→Y be a function. When does *f* possess an inverse?

For each of the following, decide if an inverse exists. If yes, find it.

1. f: N→Z defined by f(j) = -j
2. f: R→ R defined by f(x) = x5
3. g: R→ (0, $\infty )$ defined by g(x) = ex
4. h: Z → Z defined by h(j) = j + 13
5. f: (0, $\infty )$ → (0, $\infty )$ defined by f(x) = 1/x
6. G:[0, $\infty )$ → [0, $\infty )$ defined by f(x) = x2

**IV** Find a bijection from

 (i) R to R; (ii) N to Z (iii) Z to N ; (iv) [0, 5] to [7, 17]; (v) (0, 1] to R; **N** to **Q**$∩$ (0, 1)

**V** Show that a mapping G: X→ Y may also be regarded as a mapping from P(X) to P(Y).

**VI** What does it mean to say that two sets have the *same cardinality*? What does it mean to say that a set is *countably infinite*?

 **VII** Show that each of the following sets is countable:

1. The set of non-negative integers.

(b)     The set of integers greater than or equal to 13.

(c)    **Z**

(d)    The set of positive even integers.

(e)     The set of even integers.

(f)     The set of odd integers.

(g) The set of rational numbers strictly between 0 and 1.

**VIII** (a) Show that a subset of a countable set is either finite or countable.

(b) Show that if *A* and *B* are disjoint countable sets, then so is the union of *A* and *B*. What if *A* and *B* are not disjoint?

(c) Show that if *A* and *B* are countable sets, then so is the Cartesian product of *A* and *B*.

(d) Prove that a countable union of countable sets is countably infinite.

(e) Prove that the set of rational numbers strictly between 0 and 1 is countable.

(f) Demonstrate that **Q** is countable.

**IX** Show that if S is a collection of sets, then cardinality is an equivalence relation on S.

**X**  Using Cantor’s diagonal argument, prove that **R** is not countable.

 

**IX** (a) Let *X* be a set. Recall the definition of the power set, P *(X)*, of *X*.

Show that the power set of a finite set is finite. In such a case, describe the relationship between |X| and | P (X)|.

1. Let X = {a, b, c, d} and let F: X → P (X) be defined by:

F(a) = {a, c, d}, F(b) = {a, d}, F(c) = φ, F(d) = {d}

Find D\* = $\left\{j\in X| j\notin F(j)\right\}$

1. Let X = Z+ and let G: X → P(X) be defined by:

G(a) = {all prime numbers, p, such that a ≤ p ≤ 2a}

Find D\* = $\left\{j\in X| j\notin G(j)\right\}$

1. Let X = **Q** and let H: X → P (X) be defined by:

$H\left(z\right)= $ {all prime numbers, *p*, such that z ≤ p ≤ 2z}

Find D\* = $\left\{q\in X| q\notin H(q)\right\}$

1. Let X = **R** and let V: X → P (X) be defined by:

$V(a)=\left\{\begin{array}{c}\left\{0\right\} if a\leq 0 \\\left(a, a+1\right) if a\in Q^{+}\~Z \\\left[a-1, a\right] if a is a positive irrational number\\\left\{a, a+3\right\} if a\in Z^{+} \end{array}\right.$

Is V well-defined? If so, find D\* = $\left\{z\in X| z\notin V(z)\right\}$

1. Prove *Cantor’s Theorem*: X and P(X) are not of the same cardinality.

**Highly recommended:** MIT lecture notes on cardinality, 24.118 (paradox and infinity)



[Georg Ferdinand Ludwig Cantor](http://www-gap.dcs.st-and.ac.uk/~history/Mathematicians/Cantor.html) (1845 – 1918) is best known for

his discovery of transfinite numbers and the creation of Set Theory

*Lenore nodded. ‘Gramma really likes antinomies. I think this guy here (looking down at the drawing on the back of the label) ‘is the barber who shaves all and only those who do not shave themselves’.*

- David Foster Wallace, **The Broom of the System**