Math 201 Class Discussion: Mappings, continued

21 November 2019

I (Injections, Surjections, and Bijections) Which of the following functions are injective, surjective, and bijective on their respective domains. Take the domains of the functions as those whose values of x for which the function is well-defined.

a) $f(x) = x^3 - 2x + 1$

b) $f(x) = \dfrac{x+1}{x-1}$

c) $f(x) = \begin{cases} x^2 & x \le 0 \\ x+1 & x > 0 \end{cases}$

d) $f(x) = \dfrac{1}{x^2+1}$

II Let X, Y, Z be non-empty sets. Assume that F: X→Y and G: Y→Z are (well-defined) mappings. For each of the following statements give a proof or counterexample.

(a) If F and G are injective, then G∘F is injective.

(b) If F and G are surjective, then G∘F is surjective.

(c) If F and G are bijective, then G∘F is bijective.

(d) If G∘F is injective, then G is injective.

(e) If G∘F is injective, then F is injective.

(f) If G∘F is surjective, then F is surjective.

(g) If G∘F is surjective, then G is surjective.

III Let f: X→Y be a function. When does *f* possess an inverse?

For each of the following, decide if an inverse exists. If yes, find it.

(a) f: N→Z defined by f(j) = -j

(b) f: R→ R defined by $f(x) = x^5$

(c) g: R→ (0, ∞) defined by $g(x) = e^x$

(d) h: Z → Z defined by h(j) = j + 13

(e) f: (0, ∞) → (0, ∞) defined by f(x) = 1/x

(f) G:[0, ∞) → [0, ∞) defined by $f(x) = x^2$

IV Find a bijection from

(i) R to R; (ii) N to Z (iii) Z to N ; (iv) [0, 5] to [7, 17]; (v) (0, 1] to R; **N** to **Q**∩ (0, 1)

V Show that a mapping G: X→ Y may also be regarded as a mapping from P(X) to P(Y).

VI What does it mean to say that two sets have the *same cardinality*? What does it mean to say that a set is *countably infinite*?

VII Show that each of the following sets is countable:

(a) The set of non-negative integers.

(b) The set of integers greater than or equal to 13.

(c) **Z**

(d) The set of positive even integers.

(e) The set of even integers.

(f) The set of odd integers.

(g) The set of rational numbers strictly between 0 and 1.

VIII (a) Show that a subset of a countable set is either finite or countable.

(b) Show that if *A* and *B* are disjoint countable sets, then so is the union of *A* and *B*. What if *A* and *B* are not disjoint?

(c) Show that if *A* and *B* are countable sets, then so is the Cartesian product of *A* and *B*.

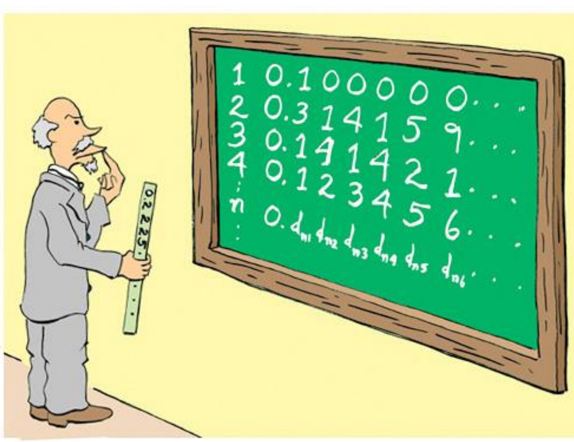
(d) Prove that a countable union of countable sets is countably infinite.

(e) Prove that the set of rational numbers strictly between 0 and 1 is countable.

(f) Demonstrate that **Q** is countable.

IX Show that if S is a collection of sets, then cardinality is an equivalence relation on S.

X Using Cantor's diagonal argument, prove that **R** is not countable.



IX (a) Let X be a set. Recall the definition of the power set, $\mathscr{P}(X)$, of X.

Show that the power set of a finite set is finite. In such a case, describe the relationship between $|X|$ and $|\mathscr{P}(X)|$.

(b) Let $X = \{a, b, c, d\}$ and let $F: X \to \mathscr{P}(X)$ be defined by:

$F(a) = \{a, c, d\}$, $F(b) = \{a, d\}$, $F(c) = \varphi$, $F(d) = \{d\}$

Find $D^* = \{j \in X | \, j \notin F(j)\}$

(c) Let $X = Z^+$ and let $G: X \to P(X)$ be defined by:

$G(a) = \{$all prime numbers, p, such that $a \le p \le 2a\}$

Find $D^* = \{j \in X | \, j \notin G(j)\}$

(d) Let $X = \mathbf{Q}$ and let $H: X \to \mathscr{P}(X)$ be defined by:

$H(z) = \{$all prime numbers, p, such that $z \le p \le 2z\}$

Find $D^* = \{q \in X | \, q \notin H(q)\}$

(e) Let $X = \mathbf{R}$ and let $V: X \to P(X)$ be defined by:

$$V(a) = \begin{cases} \{0\} \; if \; a \le 0 \\ (a, a+1) \; if \; a \in Q^+ \sim Z \\ [a-1, a] \; if \; a \; is \; a \; positive \; irrational \; number \\ \{a, a+3\} \; if \; a \in Z^+ \end{cases}$$

Is V well-defined? If so, find $D^* = \{z \in X | \, z \notin V(z)\}$

(f) Prove *Cantor's Theorem*: X and $\mathscr{P}(X)$ are not of the same cardinality.

Highly recommended: MIT lecture notes on cardinality, 24.118 (paradox and infinity)

Georg Ferdinand Ludwig Cantor (1845 – 1918) is best known for his discovery of transfinite numbers and the creation of Set Theory

Lenore nodded. 'Gramma really likes antinomies. I think this guy here (looking down at the drawing on the back of the label) 'is the barber who shaves all and only those who do not shave themselves'.

- David Foster Wallace, **The Broom of the System**