Math 201 Class Discussion: Georg Cantor

26 November 2019

I (Review) Let f: X→Y be a function. When does *f* possess an inverse?

For each of the following, decide if an inverse exists. If yes, find it.

(a) f: N→Z defined by f(j) = -j

(b) f: R→ R defined by f(x) = x^5

(c) g: R→ (0, ∞) defined by g(x) = e^x

(d) h: Z → Z defined by h(j) = j + 13

(e) f: (0, ∞) → (0, ∞) defined by f(x) = 1/x

(f) G:[0, ∞) → [0, ∞) defined by f(x) = x^2

II (Review) Find a bijection from

(i) R to R; (ii) N to Z (iii) Z to N ; (iv) [0, 5] to [7, 17]; (v) (0, 1] to R; **N** to **Q**∩ (0, 1)

III (a) Show that a subset of a countable set is either finite or countable.

(b) Show that if *A* and *B* are disjoint countable sets, then so is the union of *A* and *B*.

What if *A* and *B* are not disjoint?

(c) Show that if *A* and *B* are countable sets, then so is the Cartesian product of *A* and *B*.

(d) Prove that a countable union of countable sets is countably infinite.

(e) Prove that the set of rational numbers strictly between 0 and 1 is countable.

(f) Demonstrate that **Q** is countable.

IV What does it mean to say that two sets have the *same cardinality*? What does it mean to say that a set is *countably infinite*?

V Show that if S is a collection of sets, then cardinality is an equivalence relation on S.

VI Using Cantor's diagonal argument, prove that **R** is not countable.

VII (a) Let *X* be a set. Recall the definition of the power set, *𝒫(X)*, of *X*.

(b) Show that a mapping G: X→ Y may also be regarded as a mapping from P(X) to

Show that the power set of a finite set is finite. In such a case, describe the relationship between |X| and |𝒫(X)|.

(c) Let X = {a, b, c, d} and let F: X → 𝒫(X) be defined by:

F(a) = {a, c, d}, F(b) = {a, d}, F(c) = φ, F(d) = {d}
Find $D^* = \{j \in X |\ j \notin F(j)\}$

(d) Let X = Z^+ and let G: X → P(X) be defined by:

G(a) = {all prime numbers, p, such that a ≤ p ≤ 2a}
Find $D^* = \{j \in X |\ j \notin G(j)\}$

(e) Let X = **Q** and let H: X → $\mathscr{P}$(X) be defined by:
$H(z) =$ {all prime numbers, p, such that z ≤ p ≤ 2z}
Find $D^* = \{q \in X |\ q \notin H(q)\}$

(f) Let X = **R** and let V: X → P (X) be defined by:

$$V(a) = \begin{cases} \{0\}\ if\ a \leq 0 \\ (a, a+1)\ if\ a \in Q^+ \sim Z \\ [a-1, a]\ if\ a\ is\ a\ positive\ irrational\ number \\ \{a, a+3\}\ if\ a \in Z^+ \end{cases}$$

Is V well-defined? If so, find $D^* = \{z \in X |\ z \notin V(z)\}$

VIII Prove ***Cantor's Theorem***: X and $\mathscr{P}$(X) are not of the same cardinality.

Highly recommended: MIT lecture notes on cardinality, 24.118 (paradox and infinity)

Georg Ferdinand Ludwig Cantor (1845 – 1918) is best known for his discovery of transfinite numbers and the creation of Set Theory

Lenore nodded. 'Gramma really likes antinomies. I think this guy here (looking down at the drawing on the back of the label) 'is the barber who shaves all and only those who do not shave themselves'.

- David Foster Wallace, **The Broom of the System**