CLASS DISCUSSION: 5 NOVEMBER 2019

Relations; Equivalence relations

1. What is meant by a *relation on the set S*?
2. Determine the relation for S = {0, 1, 3, 5, 8} defined by $\leq$.
3. For the set in (2) determine the relation for S = {0, 1, 3, 5, 8} defined by $\neq$.
4. What does it mean for R to be *reflexive*? *symmetric*? *transitive*?
5. What is an *equivalence relation* on S?
6. Explain how an equivalence relation corresponds to a partition on the set S.
7. What does the term *equivalence class* mean?

8. Determine which of the three properties "reflexive," "symmetric," and "transitive" apply to each of the following relations on the set of integers, Z. For each relation that is an equivalence relation, describe the equivalence classes.

   a R b iff

   1. $a = b$ 2. $a \leq b$ 3. $a < b$, 4. $a \mid b$ 5. $|a| = |b|$

   6. $a^2 + a = b^2 + b$ 7. $a < |b|$, 8. $ab > 0$ 9. $ab \geq 0$

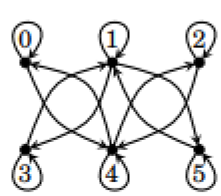
   10. $a + b > 0$ 11. $a \equiv b \mod 4$ 12. $a \equiv b \mod m$ (where $m > 0$)

9. Do the same as in (1) for the following relations on the set of all people. p R q iff
   a. p "is a father of" q
   b. p "is a sister of" q
   c. p "is a friend of" q
   d. p "is an aunt of" q
   e. p "is a descendant of" q
   f. p "has the same height" as q
   g. p "likes" q
   h. p "knows" q
   i. p "is married to" q
   j. p is a Facebook friend of q

10. Let A = { 1,2,3,4}, and consider the following set:
    S = { (1,1),(1,3),(3,1),(3,3),(2,2),(2,4),(4,2),(4,4)} $\subseteq A \times A$. Can you guess what S means?

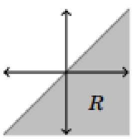
Exercises for Section 11.0

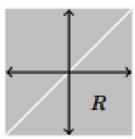
1. Let $A = \{0,1,2,3,4,5\}$. Write out the relation R that expresses $>$ on A. Then illustrate it with a diagram.
2. Let $A = \{1,2,3,4,5,6\}$. Write out the relation R that expresses $\mid$ (divides) on A. Then illustrate it with a diagram.
3. Let $A = \{0,1,2,3,4,5\}$. Write out the relation R that expresses $\geq$ on A. Then illustrate it with a diagram.
4. Here is a diagram for a relation R on a set A. Write the sets A and R.

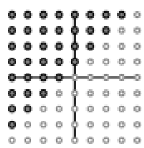


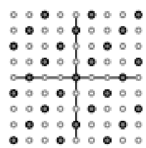
5. Consider the relation $R = \{(0,0),(\sqrt{2},0),(0,\sqrt{2}),(\sqrt{2},\sqrt{2})\}$ on $\mathbb{R}$. Is R reflexive? Symmetric? Transitive? If a property does not hold, say why.
6. Consider the relation $R = \{(x,x) : x \in \mathbb{Z}\}$ on $\mathbb{Z}$. Is this R reflexive? Symmetric? Transitive? If a property does not hold, say why. What familiar relation is this?
7. There are 16 possible different relations R on the set $A = \{a,b\}$. Describe all of them. (A picture for each one will suffice, but don't forget to label the nodes.) Which ones are reflexive? Symmetric? Transitive?
8. Define a relation on $\mathbb{Z}$ as xRy if $|x-y| < 1$. Is R reflexive? Symmetric? Transitive? If a property does not hold, say why. What familiar relation is this?
9. Let $A = \{1,2,3,4,5,6\}$. How many different relations are there on the set A?
10. Consider the subset $R = (\mathbb{R}\times\mathbb{R}) - \{(x,x) : x \in \mathbb{R}\} \subseteq \mathbb{R}\times\mathbb{R}$. What familiar relation on $\mathbb{R}$ is this? Explain.
11. Given a finite set A, how many different relations are there on A?

In the following exercises, subsets R of $\mathbb{R}^2 = \mathbb{R}\times\mathbb{R}$ or $\mathbb{Z}^2 = \mathbb{Z}\times\mathbb{Z}$ are indicated by gray shading. In each case, R is a familiar relation on $\mathbb{R}$ or $\mathbb{Z}$. State it.

12. 

13. 

14. 

15. 

Exercises for Section 11.1

1. Consider the relation $R = \{(a,a),(b,b),(c,c),(d,d),(a,b),(b,a)\}$ on set $A = \{a,b,c,d\}$. Is R reflexive? Symmetric? Transitive? If a property does not hold, say why.
2. Consider the relation $R = \{(a,b),(a,c),(c,c),(b,b),(c,b),(b,c)\}$ on the set $A = \{a,b,c\}$. Is R reflexive? Symmetric? Transitive? If a property does not hold, say why.
3. Consider the relation $R = \{(a,b),(a,c),(c,b),(b,c)\}$ on the set $A = \{a,b,c\}$. Is R reflexive? Symmetric? Transitive? If a property does not hold, say why.
4. Let $A = \{a,b,c,d\}$. Suppose R is the relation

$$R = \{(a,a),(b,b),(c,c),(d,d),(a,b),(b,a),(a,c),(c,a),(a,d),(d,a),(b,c),(c,b),(b,d),(d,b),(c,d),(d,c)\}.$$

Is R reflexive? Symmetric? Transitive? If a property does not hold, say why.
5. Consider the relation $R = \{(0,0),(\sqrt{2},0),(0,\sqrt{2}),(\sqrt{2},\sqrt{2})\}$ on $\mathbb{R}$. Is R reflexive? Symmetric? Transitive? If a property does not hold, say why.
6. Consider the relation $R = \{(x,x) : x \in \mathbb{Z}\}$ on $\mathbb{Z}$. Is R reflexive? Symmetric? Transitive? If a property does not hold, say why. What familiar relation is this?
7. There are 16 possible different relations R on the set $A = \{a,b\}$. Describe all of them. (A picture for each one will suffice, but don't forget to label the nodes.) Which ones are reflexive? Symmetric? Transitive?
8. Define a relation on $\mathbb{Z}$ as xRy if $|x-y| < 1$. Is R reflexive? Symmetric? Transitive? If a property does not hold, say why. What familiar relation is this?
9. Define a relation on $\mathbb{Z}$ by declaring xRy if and only if x and y have the same parity. Is R reflexive? Symmetric? Transitive? If a property does not hold, say why. What familiar relation is this?
10. Suppose $A \neq \emptyset$. Since $\emptyset \subseteq A\times A$, the set $R = \emptyset$ is a relation on A. Is R reflexive? Symmetric? Transitive? If a property does not hold, say why.
11. Suppose $A = \{a,b,c,d\}$ and $R = \{(a,a),(b,b),(c,c),(d,d)\}$. Is R reflexive? Symmetric? Transitive? If a property does not hold, say why.
12. Prove that the relation $|$ (divides) on the set $\mathbb{Z}$ is reflexive and transitive. (Use Example 11.8 as a guide if you are unsure of how to proceed.)
13. Consider the relation $R = \{(x,y) \in \mathbb{R}\times\mathbb{R} : x - y \in \mathbb{Z}\}$ on $\mathbb{R}$. Prove that this relation is reflexive, symmetric and transitive.

14. Suppose R is a symmetric and transitive relation on a set A, and there is an element $a \in A$ for which aRx for every $x \in A$. Prove that R is reflexive.
15. Prove or disprove: If a relation is symmetric and transitive, then it is also reflexive.
16. Define a relation R on $\mathbb{Z}$ by declaring that xRy if and only if $x^2 \equiv y^2 \pmod 4$. Prove that R is reflexive, symmetric and transitive.

Exercises for Section 11.2

1. Let $A = \{1,2,3,4,5,6\}$, and consider the following equivalence relation on A: $R = \{(1,1),(2,2),(3,3),(4,4),(5,5),(6,6),(2,3),(3,2),(4,5),(5,4),(4,6),(6,4),(5,6),(6,5)\}$ List the equivalence classes of R.
2. Let $A = \{a,b,c,d,e\}$. Suppose R is an equivalence relation on A. Suppose R has two equivalence classes. Also aRd, bRc and eRd. Write out R as a set.
3. Let $A = \{a,b,c,d,e\}$. Suppose R is an equivalence relation on A. Suppose R has three equivalence classes. Also aRd and bRc. Write out R as a set.
4. Let $A = \{a,b,c,d,e\}$. Suppose R is an equivalence relation on A. Suppose also that aRd and bRc, eRa and cRe. How many equivalence classes does R have?
5. There are two different equivalence relations on the set $A = \{a,b\}$. Describe them. Diagrams will suffice.
6. There are five different equivalence relations on the set $A = \{a,b,c\}$. Describe them all. Diagrams will suffice.
7. Define a relation R on $\mathbb{Z}$ as xRy if and only if $3x-5y$ is even. Prove R is an equivalence relation. Describe its equivalence classes.
8. Define a relation R on $\mathbb{Z}$ as xRy if and only if x^2+y^2 is even. Prove R is an equivalence relation. Describe its equivalence classes.
9. Define a relation R on $\mathbb{Z}$ as xRy if and only if $4 \mid (x+3y)$. Prove R is an equivalence relation. Describe its equivalence classes.
10. Suppose R and S are two equivalence relations on a set A. Prove that $R \cap S$ is also an equivalence relation. (For an example of this, look at Figure 11.2. Observe that for the equivalence relations R_2, R_3 and R_4, we have $R_2 \cap R_3 = R_4$.)
11. Prove or disprove: If R is an equivalence relation on an infinite set A, then R has infinitely many equivalence classes.
12. Prove or disprove: If R and S are two equivalence relations on a set A, then $R \cup S$ is also an equivalence relation on A.