MATH 201: PREPARING FOR TEST 2
17 OCTOBER 2019

Chapter 3 (sections 1 – 5, 7 (special case), 8 – 10);
Chapter 10 (sections 1 & 2);
Chapters 4 – 6 (as much as we cover on Tuesday)

Albrecht-Dürer, **Melancholia**

(Note the magic square in the background.)

- Types of problems:
  - counting: permutations, combinations, stars & bars
  - proofs (direct, cases, induction (ordinary & strong), contrapositive, contradiction, if and only if, conditional)
  - definitions in the case of number theory (odd, even, divisibility, prime number, etc.)
  - fill in the blank
  - True/False
  - find a counter-example
  - given proof with a missing part, fill in the missing part
  - given a false "proof," correct it
  - inclusion/exclusion principle (special case)

Practice problems:

1. Consider the set of all binary sequences of length 10. How many such strings exist if

(a) No condition?

(b) Exactly three 1s?

(c) First digit and last digit must be 0?

(d) The first digit or last digit must be 0?

(e) The sum of the digits is 7?

2. Let n be an integer. Prove that $n^3 + n^2 + n$ is even if and only if n is even.

3. Given integers, p and q prove that if both pq and p+q are even, then both *p* and *q* are even using

(a) Proof by contrapositive

(b) Direct proof

4. Given integers c and d, where $c \geq 2$, prove, using the method of contradiction, that either $c \nmid d$ or $c \nmid (d+1)$.

5. Let *a* and *b* be non-zero integers. Then we say write a|b if __________________________.

6. The two basic steps in a proof by induction are called:

(A) ___________________________________

(B) ____________________________________

7. Explain why every integer can be expressed in the form 5n, 5n+1, 5n+2, 5n+3 or 5n+4.

8. Find the flaw in the following "proof":

(from A. J. Hildebrand, notes from University of Illinois)

Claim: For all $n \in \mathbb{N}$, $(*)\ \sum_{i=1}^{n} i = \frac{1}{2}(n+\frac{1}{2})^2$

Proof: We prove the claim by induction.

Base step: When $n = 1$, $(*)$ holds.

Induction step: Let $k \in \mathbb{N}$ and suppose $(*)$ holds for $n = k$. Then

$$\begin{aligned}\sum_{i=1}^{k+1} i &= \sum_{i=1}^{k} i + (k+1)\\ &= \frac{1}{2}\left(k+\frac{1}{2}\right)^2 + (k+1) \quad \text{(by ind. hypothesis)}\\ &= \frac{1}{2}\left(k^2+k+\frac{1}{4}+2k+2\right) \quad \text{(by algebra)}\\ &= \frac{1}{2}\left(\left(k+1+\frac{1}{2}\right)^2 - 3k - \frac{9}{4} + k + \frac{1}{4} + 2k + 2\right) \quad \text{(more algebra)}\\ &= \frac{1}{2}\left((k+1)+\frac{1}{2}\right)^2 \quad \text{(simplifying)}.\end{aligned}$$

Thus, $(*)$ holds for $n = k+1$, so the induction step is complete.

Conclusion: By the principle of induction, $(*)$ holds for all $n \in \mathbb{N}$.

9. Explain Euclid's argument (by contradiction) that there are infinitely many primes.

10. Prove, using strong induction, that every integer greater than 1 can be factored into primes.

11. In class, we proved that, for all natural numbers *n*, any $2^n \times 2^n$ punctured chessboard could be tiled by tri-ominos.

Prove independently that 3 must be a divisor of $2^{2n} - 1$

12. Albertine is teaching Chem 105 this semester. She has a total of 100 students in her class. Taking a survey (via Piazza), she finds that 28 students like dogs, 26 like cats, and 16 like rats. There are 12 students who like cats and dogs, 4 who like dogs and rats, and 6 who like cats and rats. Only 2 students like dogs, cats, and rats. How many students do not like any of the 3 animals: dogs, cats, rats.


13.  Three *distinguishable* dice are thrown. In how many ways can the *maximum* of the 3 numbers occurring equal 5?

14. If you are dealt a hand of 7 cards from a standard deck (without regard to order), how many ways can you have:

(a) A flush? (all 7 of the same suit)
(b) 3 pairs (excluding 4 of a kind)?
(c) 4 of a kind and one pair (excluding 3 of a kind)?
(d) 4 of a kind and no other pair?
(e) No pair at all?

15. Find the flaw in the following bogus proof. Explain!

- Step 1: Let *a* = *b*.
- Step 2: Then $a^2 = ab$
- Step 3: Then $a^2 + a^2 = a^2 + ab$
- Step 4: Then $2a^2 = a^2 + ab$
- Step 5: Then $2a^2 – 2ab = a^2 + ab – 2ab$
- Step 6: Then $2a^2 – 2ab = a^2 – ab$
- Step 7: This can be written as $2(a^2 – ab) = 1(a^2 – ab)$
- Step 8: Dividing each side by $(a^2 – ab)$ yields $1 = 2$

16. Consider a group of 11 American students {A, B, C, …, J, K} visiting the Louvre. In front of La Giaconda, they

line up at random for a photograph to be taken by their tour guide. In how many ways can the 11 students line up so that B and C are side-by-side?

17. (a) If a Martian has an infinite number of red, blue, yellow, and black socks in a drawer, how many socks must the Martian pull out of the drawer to guarantee it has a pair?

(b) A box contains 6 red, 8 green, 10 blue, 12 yellow, and 15 white balls. What is the minimum no. of balls we have to choose randomly from the box to ensure that we get 9 balls of same color?

18. Consider the equation $x_1 + x_2 + \ldots + x_5 = 99$

(a) How many solutions are there in non-negative integers?

(b) How many solutions are there in positive integers?

19. Prove that if x and y are irrational, must x + y or xy be irrational as well?

Gambling was in his blood. Couldn't help it – he had an irregular heartbeat. Even his pulse was impulsive.