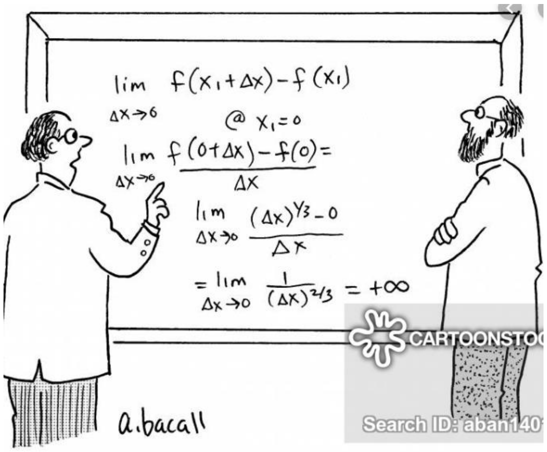
# MATH 201 PRACTICE PROBLEMS FOR TEST III

## (REVISED)




"I proved that when you start to count your blessings, you find that they're infinite."

**1.** For any real numbers, *c* and *d*, let us define the binary operation ☺ as follows:

$$c ☺ d = c^2 + d^2 - 1$$

Give either a *brief* justification or counterexample for each of the following assertions:

(a) The set of integers is closed under the operation ☺ .

(b) The set of even integers is closed under the operation ☺ ?

(c) The set of odd integers is closed under ☺.

(d) The set of positive integers is closed under ☺.

(e) The set of rational numbers is closed under ☺.

(f) The set of irrational numbers is closed under ☺.

**2.** Let S be the set of all polynomials for which odd powers of x (i.e., 1, 3, 5, 7,…) do not appear.

For example, $4 + \frac{1}{2} x^2 - \pi x^8 \in S$, *but* $x^2 - \frac{5}{4}x^{81} \notin S$.

(a) Is S closed under the operation of differentiation? Why?

(b) Is S closed under the operation of taking the second derivative?

(c) Is S closed under the operation multiplication by $x^3$?

(d) Is S closed under the operation multiplication by $x^4$?

(e) Is S closed under the following unary operation, ● ?

For all $p \in S$ *define* $\Pi p$ *as follows*: $\Pi p = p(1 + x^4)$

Note: $\Pi p$ *means* that polynomial p is evaluated at $1 + x^4$

(f) Is S closed under the following binary operation ☹?

For all $p \in S$ $p☹ = p(1 + x)$ Here as before, we interpret p(1 + x) is the polynomial p evaluated at 1 + x.

**3.** Define the following binary relation R on **N**. For c, d ∈ $\mathbf{Z}^+$, cRd if $|c - d| < 5$. *(Justify each answer!)*

(a) Is R reflexive? Why?
(b) Is R symmetric? Why?
(c) Is R transitive? Why?

**4.** Define the binary relation $R$ on a non-empty set, S, of books as follows:

For a, b $\in$ S, a$R$b if book ***a*** costs more **and** contains fewer pages than book ***b***. *(Justify each answer!)*

(a) Is R reflexive? Why?
(b) Is R symmetric? Why?
(c) Is R transitive? Why?

**5.** Again, let $S$ be a non-empty set of books. Let $H$ be the binary relation defined by:
for $a$, $b \in S$, a$H$b if book ***a*** costs more ***or*** contains fewer pages than book ***b***. *(Justify each answer!)*
(a) Is $H$ reflexive? Why?
(b) Is $H$ symmetric? Why?
(c) Is $H$ transitive? Why?

**6.** Let $x$ and $y$ be real numbers. Prove that if $x^2 + 5y = y^2 + 5x$. Prove that either x = y or x + y = 5.

**7.** Using the *Euclidean algorithm*, find the gcd of 7701543 and 3141

**8.** Using the method of contrapositive proof, prove each of the following statements:

(a) Suppose that x, y $\in$ Z. If $x^2(y + 3)$ is even, prove that x is even or y is odd.
(b) Let a $\in$ Z. Prove that if $a^2$ is not divisible by 4, then a is odd.

**9.** Prove that there exist no integers a and b for which 390a + 63b = 1.

**10.** Use Fermat's little theorem to show that 17 divides $11^{104}+1$.

**11.** Prove that $\sqrt{6}$ is irrational.

**12.** Let a $\in$Z. Prove that $a^3 \equiv a \pmod 3$.

**13.** Let a, b $\in$**Z**. Prove that $(a – 3)b^2$ is even if and only if a is odd and b is even.

**14.** The three most recent appearances of Haley's comet were in the years 1835, 1910, and 1986.
The next occurrence will be in 2061. Prove that $1835^{1910} + 1986^{2061} \equiv 0 \pmod 7$.

**15.** Find $2013^{13}$ (mod 13)

**16.** Let W be the set of all words in the 2019 edition of the Oxford English dictionary. Define xRy if x and y have at least one letter in common. Is R reflexive? Symmetric? Transitive?

**17.** Let A= $R^3$. Let a~b mean that a and b have the same z coordinate. Does this define an equivalence relation?

**18.** Let S be a finite set and A=$\mathcal{P}$(S), the power set of S. For any a,b$\in$A, let a~b mean that a and b have the same cardinality (that is the same number of members).

Show that ~ is an equivalence relation. Compute the equivalence classes when S ={1,2,3}.

**19.** The following purports to prove that the reflexivity condition is unnecessary, that is, it can be derived from symmetry and transitivity:

Suppose a~b. By symmetry, b~a. Since a~b and b~a, by transitivity, a~a. Therefore, ~ is reflexive.

What's wrong with this argument?

**20.** True or False? Explain.

(a) If $R$ is an equivalence relation on a finite non-empty set A, then the equivalence classes of $R$ all have the same number of elements.

(b) Let $R$ be a relation defined on the set **Z** by aRb if $a \neq b$. Then R is symmetric and transitive.

**21.** Define GCD. Prove that gcd(n, m) = gcd(n + m, m).

**22.** Show that gcd(n, m) = gcd(n − m, m).

**23.** Give an example to show that gcd(n, m) = gcd(n + m, n − m) need not be true.

**24.** Suppose that n is even and gcd(n, m) = 5. Show that m is odd.

**25.** Find all numbers k with $0 \leq k \leq 100$ such that gcd(100, k) = 5.

**26.** (a) Show that there is no integer x satisfying the equation $2x + 1 = 5x - 4$

(b) Show that there is no integer x satisfying the equation $18x^2 + 39x - 7 = 0$

(c) Show that the system of equations

$$11x - 5y = 7$$
$$9x + 10y = -3$$

has no integer solution.

**27.** Let A = {1, 2, 3, 4, 5}, and let

$R$ = {(1, 1), (1, 3), (1, 4), (2, 2), (2, 5), (3, 1), (3, 3), (3, 4), (4, 1), (4, 3), (4, 4), (5, 2), (5, 5)} define an equivalence relation on A. (You may wish to check this!) Which of the following is an *equivalence class*?

a) {1, 2, 3} b) {2, 3, 5} c) {1, 3, 4} d) {1, 2} e) {1, 2, 3, 4, 5}

**28.** State Euclid's lemma. Prove that gcd(a, bc) = 1 if and only if gcd(a, b) = 1 and gcd(a, c) = 1.

**29.** Let X denote the set of all continuous functions on [0, 1].

(a) Is X closed under addition, subtraction, multiplication, division?

(b) Is X closed under composition? Differentiation?

**30.** Let A be the set of all polynomials in the variable x. Is A closed under:

(a) Addition, subtraction, multiplication, composition, differentiation?

(b) Consider the unary operations in A defined by, for $p \in A$

$(i)\ U(p) = p(2x)$; (ii) $U(p) = p''(x) + 4p'(x) + xp(x)$. (iii) $U(p) = x^2p(x)$

Under which of these operations is A closed?

**31.** (a) Prove that $\log_2 5$ is irrational.

(b) Find $2^{2019}$ mod 17

(c) Find $2^{2019}$ mod 33.

(d) Find the units and ten digits of (984321)(454443)(444555).

**32.** Let Y be the power set of a finite non-empty set F.

(a) Is Y closed under complements?

(b) Is Y closed under union? Intersection?