MATH 201 TEST III 14 NOVEMBER 2019

Name: ________________________

"Euclid may have been the first to define primality in his Elements approximately 300 BC...He realized that the even perfect numbers are all closely related to the primes of the form 2^p-1 for some prime p (now called Mersennes). So the quest for these jewels began near 300 BC."

- C. Caldwell, from The Prime Pages

"Mathematicians have tried in vain to this day to discover some order in the sequence of prime numbers, and we have reason to believe that it is a mystery into which the mind will never penetrate."

- Leonhard Euler

PART I *[3 pts each]*: Complete each sentence below:

1. Let p and q be integers and let m be an integer greater than 1. Then we write $p \equiv q \pmod{m}$ if

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2. An *equivalence relation* on a set S *is* a binary operation, R, on S that has the properties:

(i) $a\mathrm{R}a$ for all $a \in S$

(ii) If aRb then bRa

(iii) If______________________________

3. State Euclid's lemma: If a, b are ____________and c is _____________then,

____

4. Let p and q be integers and let m be an integer greater than 1. We write $p \equiv q \pmod{m}$ if

____

5. Let *p* be __________ and *q be* ___________. Then we say write a|b if __________________________.

6. Let *n* and *d* be integers with $d \neq 0$. The *division algorithm* states that there exist unique integers q and r such that ___________________________ where _______________________.

7. 15^{99999} is congruent to _______________ (mod 16).

8. If $a \equiv 9 \pmod{11}$ and $b \equiv 7 \pmod{11}$, then $3a + 5b \equiv$ _____________ (mod 11)

9. Let S be a set. Then an equivalence relation on S corresponds to a ____________ on S and vice-versa.

10. Using the Euclidean algorithm, compute gcd(91091, 7007)

11. Write the contrapositive of the following statement:

“If the sidewalk is not slippery, then it did not snow last night”

12. Compute 24^{39} mod 7.

13. Give an example to show that gcd(n, m) = gcd(n + m, n − m) need not be true

Part II *[8 pts each]* Choose any 10 of the following 12 problems. You may answer more than 10 to earn extra credit.

1. (i) Consider the following relation on N.

a R b if a|b or b|a.

(a) Is R reflexive?

(b) Is R symmetric?

(c) Is R transitive?

(d) If R is an equivalence relation, find the equivalence class whose representative is 6.

(ii) Consider the following relation on Z:

p ~ q if |p| = |q|

(a) Is ~ reflexive?

(b) Is ~ symmetric?

(c) Is ~ transitive?

(d) If ~ is an equivalence relation, find the equivalence class whose representative is 6.

2. Let $m \geq 2$ be an integer. The converse to Fermat's little theorem states:

If $a^{m-1} \equiv 1 \pmod{m}$, it need not follow that m is prime.

Give a counter-example to show that this converse need not be true.

3. Using your method of choice, prove that if n is an integer, then 4 $\mathit{is\ not\ a\ divisor\ of}\ n^2 + 2$.

4. *[2 pts each]* Let $S = N \times N$. Define the following relation, ☼, on *S*:

(a, b) ☼ (c, d) if ad = bc.

(a) Is ☼ *reflexive*? Why?

(b) Is ☼ *symmetric*? Why?

(c) Is ☼ *transitive*? Why?

(d) Is ☼ an *equivalence relation* on *S?* Why?

5. Prove that $\sqrt[3]{2}$ is irrational.

6. Consider the following binary operations on **N**:

(i) For a, b ∈ **N** $then\ a \nabla b = ab + a - b$.

(ii) For a, b ∈ **N** $then\ a \otimes b = ab + a + b$.

(iii) For a, b ∈ **N** $then\ a \blacksquare b = (a + b)^2$.

(iv) For a, b ∈ **N** $then\ a$ ☺ $b = \max\{a, b\}$

(a) Under which of the four binary operators above is **N** closed? (Give all that are applicable, if any.)

Answer: ______________________________

(b) Which of the four operators above are commutative (i.e., a *OP* b = b *OP* a)?

Answer: ______________________________

(c) Which of the four operators above are associative, that is,
(a *OP* b) *OP* c = a *OP* (b *OP* c) ?

Answer: ______________________________

7. Compute $2^{20} + 3^{30} + 4^{40} + 5^{50} + 6^{60} \pmod 7$

8. Define a relation ⋕ on **Z** as follows: a⋕b means $|a - b| \leq 7$.

Is ⋕ *reflexive*? Why?

Is ⋕ *symmetric*? Why?

Is ⋕ *transitive*? Why?

If ⋕ $is\ an\ equivalence\ relatoin, find\ the\ equivalence\ class\ represented\ by$ 0.

9. The following purports to prove that the reflexivity condition is unnecessary, that is, it can be derived from symmetry and transitivity:

Suppose $a \sim b$*. By symmetry,* $b \sim a$*. Since* $a \sim b$ *and* $b \sim a$*, by transitivity,* $a \sim a$*. Therefore,* $\sim$ *is reflexive.*

What is wrong with this argument?

10. Find a *counterexample* to the following false statement:

Let a, b, c, and $n \geq 2$ be integers. Assume that $c \not\equiv 0 \ (mod\ n)$.

Then, if $ca \equiv cd \ (mod\ n)$, it follows that $a \equiv b \ (mod\ n)$.

11. (a) Show that there is no integer x satisfying the equation $28x^2 + 35x - 9 = 0$

(b) Show that the system of equations

$$12x - 7y = 7$$
$$8x + 12y = -3$$

has no integer solution.

12. Let a, b $\in$ **Z**. Prove that $(a + 5)b^2$ is even if and only if a is odd or b is even.

EXTRA CREDIT:

Prove that $x^2 - 7y = 3$ has no solution (in integers).