MATH 201: PREPARING FOR TEST 2

Chapter 3 (sections 1 – 5, 7 (special case), 8 – 10);
Chapter 10 (section 1);
Chapters 4 – 7 (as much as we cover Monday & Wednesday)

Albrecht-Dürer, **Melencholia**

(Note the magic square in the background.)

- Types of problems:
  - counting: permutations, combinations, stars & bars
  - proofs (direct, cases, induction (ordinary), contrapositive, contradiction, if and only if, conditional)
  - definitions in the case of number theory (odd, even, divisibility, prime number, etc.)
  - statements of theorems (pigeon-hole principle, inclusion/exclusion, division algorithm, etc.)
  - fill in the blank
  - True/False
  - find a counter-example
  - given proof with a missing part, fill in the missing part
  - given a false "proof," correct it
  - modular arithmetic
  - pigeon-hole principle
  - inclusion/exclusion principle (special case)

Practice problems:

1. Consider the set of all binary sequences of length 10. How many such strings exist if
   - (a) No condition?
   - (b) Exactly three 1s?
   - (c) First digit and last digit must be 0?
   - (d) First digit or last digit must be 0?
   - (e) The sum of the digits is 7?

2. Let n be an integer. Prove that $n^3 + n^2 + n$ is even if and only if n is even.
3. Given integers, p and q prove that if both pq and p+q are even, then both a and b are even using
   - (a) Proof by contrapositive
   - (b) Direct proof
4. Given integers c and d, where c ≥ 2, prove, using the method of contradiction, that either $c \nmid d$ or $c \nmid (d+1)$.
5. Let p and q be integers and let m be an integer greater than or equal to 1. Then we write p ≡ q (mod m) if

   ____

6. The basic version of the pigeonhole principle states that if there are n-pigeon holes, k pigeons and ________ then

   ____.

7. Let a and b be non-zero integers. Then we say write a|b if _________________________.

8. Let n and d be integers with d ≠ 0. The *division algorithm* states that there exist unique integers q and r such that

   ___________________________ where _______________________.

9. The two basic steps in a proof by induction are called:
   - (A) __________________________________
   - (B) ___________________________________

10. 15^{99999} (mod 16) is congruent to _______________

11. If a ≡ 9 (mod 11) and b ≡ 7 (mod 11), then 3a + 5b ≡ _____________ (mod 11)

12. Explain why every integer can be expressed in the form 5n, 5n+1, 5n+2, 5n+3 or 5n+4.
13. Find the units digit of 17^{902}. (*Hint:* Think mod 10.)

14. There are 800,000 pine trees in a forest. Each pine tree has no more than 600,000 needles. Prove that at least two trees have the same number of needles.

15. Harvey Swick Middle School has 1,837 pupils currently registered.

    Prove that at least 6 of the students celebrate their birthdays on the same day of the year.

16. Let a, b, c, d and m be integers, with $m \geq 2$. Prove that if $a \equiv b \pmod{m}$ and $c \equiv d \pmod{m}$

17. Find $1! + 2! + 3! + \ldots + 12345! \pmod{12}$

18. Let a and c be *positive* integers and let $m \geq 2$ be an integer. Demonstrate, providing a counter-example, that if $ca \equiv cb \pmod{m}$ then it needn't follow that $a \equiv b \pmod{m}$.

19. Let a, b and c be positive integers. Prove that if $ca|cb$ then $a|b$.

20. Find the flaw in the following "proof":

(from A. J. Hildebrand, notes from University of Illinois)

Claim: For all $n \in \mathbb{N}$, $(*)\ \sum_{i=1}^{n} i = \frac{1}{2}(n + \frac{1}{2})^2$

Proof: We prove the claim by induction.

Base step: When $n = 1$, $(*)$ holds.

Induction step: Let $k \in \mathbb{N}$ and suppose $(*)$ holds for $n = k$. Then

$$\begin{aligned}
\sum_{i=1}^{k+1} i &= \sum_{i=1}^{k} i + (k+1) \\
&= \frac{1}{2}\left(k + \frac{1}{2}\right)^2 + (k+1) \quad \text{(by ind. hypothesis)} \\
&= \frac{1}{2}\left(k^2 + k + \frac{1}{4} + 2k + 2\right) \quad \text{(by algebra)} \\
&= \frac{1}{2}\left(\left(k + 1 + \frac{1}{2}\right)^2 - 3k - \frac{9}{4} + k + \frac{1}{4} + 2k + 2\right) \quad \text{(more algebra)} \\
&= \frac{1}{2}\left((k+1) + \frac{1}{2}\right)^2 \quad \text{(simplifying).}
\end{aligned}$$

Thus, $(*)$ holds for $n = k + 1$, so the induction step is complete.

Conclusion: By the principle of induction, $(*)$ holds for all $n \in \mathbb{N}$.

21. Prove that $111^{333} + 333^{111}$ is divisible by 7.

Fifteen students in French 103 were given a dictation quiz. Albertine made 13 errors. Each of the other students made fewer errors. Prove that *at least two* students made the same number of errors. *(Who are the pigeons and what are the pigeon holes?)*

22. In class, we proved that, for all natural numbers *n*, any $2^n \times 2^n$ punctured chessboard can be tiled by tri-ominos. Prove independently that 3 must be a divisor of $2^{2n} - 1$

23. Find the flaw in the following "proof":

(from A. J. Hildebrand, notes from University of Illinois)

Claim: For every nonnegative integer n, $(*)\ 2^n = 1$.

Proof: We prove that $(*)$ holds for all $n = 0, 1, 2, \ldots$, using strong induction with the case $n = 0$ as base case.

Base step: When $n = 0$, $2^0 = 1$, so $(*)$ holds in this case.

Induction step: Suppose $(*)$ is true for all integers n in the range $0 \le n \le k$, i.e., assume that for all integers in this range $2^n = 1$. We will show that $(*)$ then holds for $n = k+1$ as well, i.e., that $(**)\ 2^{k+1} = 1$.

We have

$$\begin{aligned} 2^{k+1} &= \frac{2^{2k}}{2^{k-1}} \quad \text{(by algebra)} \\ &= \frac{2^k \cdot 2^k}{2^{k-1}} \quad \text{(by algebra)} \\ &= \frac{1 \cdot 1}{1} \quad \text{(by strong ind. hypothesis applied to each term)} \\ &= 1 \quad \text{(simplifying)}, \end{aligned}$$

proving $(**)$. Hence the induction step is complete.

Conclusion: By the principle of strong induction, $(*)$ holds for all nonnegative integers n.

24. Using modular arithmetic, find the remainder when

(a) 2^{125} is divided by 7.

(b) (12)(29)(408) is divided by 13

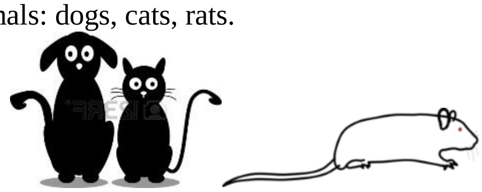
(c) 7^{1942} is divided by 5.

Restate each of the above as a statement in modular arithmetic.

25. (a) If it is now 2:00, what time would it be in 12345 hours?

(b) Is $2222^{5555} + 5555^{2222}$ divisible by 7?

26. Albertine is teaching Chem 105 this semester. She has a total of 100 students in her class. Taking a survey (via Piazza) she finds that 28 students like dogs, 26 like cats, and 16 like rats. There are 12 students who like cats and dogs, 4 who like dogs and rats, and 6 who like cats and rats. Only 2 students like dogs, cats and rats. How many students do not like any of the 3 animals: dogs, cats, rats.



27.  Three *distinguishable* dice are thrown. In how many ways can the *maximum* of the 3 numbers occurring equal 5?

28. IF you are dealt a hand of 7 cards from a standard deck (without regard to order), how many ways can you have:

(a) A flush?

(b) 3 pairs (excluding 4 of a kind)?

(c) 4 of a kind and one pair (excluding 3 of a kind)?

(d) 4 of a kind and no other pair?

(e) No pair at all?

29. Prove that $2^{1/3}$ is irrational.