The non-multiplicativity of the signature of a fibre bundle and its relation to asymmetric *L*-theory, *SK* groups and *TQFT*

Carmen Rovi

University of Edinburgh

Young Topology Meeting
Imperial College London
25-26 April 2013.

Outline

1. Statement of the problem
2. Methods which have been used
3. Previous results
4. Algebraic approach
5. Relations to TQFT and signatures of Lefschetz fibrations

Statement of the problem and methods

The problem

Let $F^n \longrightarrow E^{n+m} \longrightarrow B^m$ be a fibre bundle with $n + m \equiv 0 \pmod 4$,
Problem: What is the relation between the signatures

$$\sigma(E), \sigma(F), \sigma(B) \in \mathbb{Z} \ ?$$

Methods

- Spectral sequences
- Atiyah-Singer index theory
- Characteristic classes
- Group cohomology (cocycles)
- Algebraic K-theory
- Algebraic L-theory

The signature

Definition

The signature of a closed oriented n-dimensional manifold M^n is denoted by $\sigma(M) \in \mathbb{Z}$.

- If $n = 4k$ then $\sigma(M)$ is defined to be the number of positive eigenvalues minus the number of negative eigenvalues of the non-singular symmetric intersection form $(H^{2k}(M;\mathbb{R}), \phi)$, where

$$\phi : H^{2k}(M;\mathbb{R}) \times H^{2k}(M;\mathbb{R}) \longrightarrow \mathbb{R}; (u, v) \mapsto \langle u \cup v, [M] \rangle .$$

- If $n \neq 4k$ then $\sigma(M) = 0 \in \mathbb{Z}$.

The non-multiplicativity of the signature

For an untwisted product of spaces X and Y the signature is multiplicative:

$$\sigma(X \times Y) = \sigma(X)\sigma(Y) \in \mathbb{Z}$$

In a fibre bundle $F \to E \to B$, the signature of the total space may not be the product of the signatures of the base space and the fibre, so that in general,

$$\sigma(E) \neq \sigma(B)\sigma(F) \in \mathbb{Z}$$

Sufficient conditions for multiplicativity

Theorem (Chern, Hirzebruch and Serre, 1957)

Let $F \to E \to B$ be a fibre bundle. *If the fundamental group* $\pi_1(B)$ *acts trivially on* $H^*(F, \mathbb{Q})$ then

$$\sigma(E) = \sigma(B)\sigma(F) \in \mathbb{Z}. \quad \textit{(used spectral sequences)}$$

Example

Atiyah 1969 and Kodaira 1967, constructed non-multiplicative examples of fibre bundles $F \to E \to B$ with $\pi_1(B)$ acting non-trivially on $H^*(F, \mathbb{Q})$: The total space E is a 4-manifold which arises as a complex algebraic surface, and B and F are compact oriented surfaces of genus 129 and 6 respectively,

$$\sigma(E) = 2^8 \neq \sigma(B)\sigma(F) = 0 \in \mathbb{Z}. \quad \textit{(used index theory)}$$

More recent examples: Lefschetz fibrations $F^2 \to E^4 \to S^2$

Multiplicativity mod 4

Multiplicativity mod 4 for surface bundles. (Meyer, 1973)

Theorem If $F^2 \to E^4 \to B^2$ is a surface bundle then

$$\sigma(E) \equiv \sigma(B)\sigma(F) \pmod 4$$ *(used group cohomology)*

Multiplicativity mod 4. (Hambleton, Korzeniewski, Ranicki, 2007)

Theorem Let $F \to E \to B$ be a fibre bundle of closed, connected, compatibly oriented manifolds of any dimension then,

$$\sigma(E) \equiv \sigma(B)\sigma(F) \pmod 4$$ *(used algebraic K-theory)*

Algebraic cobordism groups I

Motivation

Two closed n-dimensional manifolds M and M' are cobordant if there exists a manifold W^{n+1} such that $\partial W = M \sqcup M'$. Cobordism groups of manifolds are denoted by Ω_n.

Various algebraic L-theoretic groups can be defined in a similar way. Here we will review the definitions of:

- symmetric L-groups
- quadratic L-groups
- visible symmetric L-groups
- asymmetric $LAsy$-groups

Algebraic cobordism groups II

- An algebraic symmetric Poincaré complex (C, ϕ) over a ring with involution A is an A-module chain complex C with symmetric Poincaré duality $\phi \simeq \phi^* : C^{n-*} \simeq C$.
  The **symmetric L-groups** are cobordism groups of algebraic symmetric Poincaré complexes,
  $$(C, \phi) \in L^n(A) \ ,$$
  generalized Witt groups of symmetric forms. The symmetric signature $\sigma^*(X) \in L^n(\mathbb{Z}[\pi_1(X)])$ of a geometric Poincaré complex X is the cobordism class of the symmetric Poincaré complex $(C(\widetilde{X}), \phi)$, with $\tilde{X} =$ universal cover of X.
- The **quadratic L-groups** are the cobordism groups of algebraic quadratic Poincaré complexes,
  $$(C, \psi) \in L_n(A)$$
  These are the Wall surgery obstruction groups, generalized Witt groups of quadratic forms.

Algebraic cobordism groups III

- The **visible symmetric L-groups** $VL^n(B)$ (Weiss, Ranicki, 1992) are the cobordism groups of visible symmetric Poincaré complexes over a space B. For the classifying space $K(\pi,1)$ of a group π, $VL^{4k}(K(\pi,1))$ is the Witt group of nonsingular symmetric forms over $\mathbb{Z}[\pi]$ with diagonal entries of the type

$$\sum_{g\in\pi} a_g(g+g^{-1}) + b \in \mathbb{Z}[\pi], \ \ (a_g \in \mathbb{Z}, b = 0, 1)$$

The forgetful map

$$L_n(\mathbb{Z}[\pi]) \longrightarrow VL^n(K(\pi,1)); (C,\psi) \longmapsto (C,(1+T)\psi)$$

is an isomorphism modulo 8 torsion.

Algebraic cobordism groups IV

- The **asymmetric** ***LAsy*-groups** are the cobordism groups of asymmetric Poincaré complexes,

$$(C, \lambda : C^{n-*} \simeq C) \in LAsy^n(A)$$

The asymmetric signature, $\sigma Asy^*(X)$ is the cobordism class of the $\mathbb{Z}[\pi_1(X)]$-module chain complex (C, λ).

It is important to note that the forgetful map

$$VL^n(K(\pi, 1)) \longrightarrow LAsy^n(\mathbb{Z}[\pi])$$

is far from begin an isomorphism, not even rationally.
Note that $VL^0(\{*\})$ is finitely generated, whereas $LAsy^0(\mathbb{Z})$ is infinitely generated.

Transfer maps in L-theory

For a fibre bundle $F^m \longrightarrow E^{n+m} \xrightarrow{p} B^n$ there exist transfer maps:

$$p^! : L_n(\mathbb{Z}[\pi_1(B)]) \longrightarrow L_{n+m}(\mathbb{Z}[\pi_1(E)])$$
$$p^! : VL^n(B) \longrightarrow VL^{n+m}(E)$$
$$p^! : LAsy^n(\mathbb{Z}[\pi_1(B)]) \longrightarrow LAsy^{n+m}(\mathbb{Z}[\pi_1(E)])$$

The transfer maps use chain level parallel transport.

$$p^! = (C(\tilde{F}), \alpha, U)^!$$

with

- $C(\tilde{F})$ is a $\mathbb{Z}[\pi_1(E)]$-module chain complex and $\tilde{F}$ is the pullback from the universal cover $\tilde{E}$ of E,
- $\alpha : C(\tilde{F}) \longrightarrow C(\tilde{F})^{n-*}$,
- $U : \mathbb{Z}[\pi_1(B)] \longrightarrow H_0(\mathrm{Hom}_{\mathbb{Z}[\pi_1(E)]}(C(\tilde{F}), C(\tilde{F})))$. U is determined by the fibre transport and encodes the information about the action of $\pi_1(B)$ on the homotopy of the fibre F.

Multiplicativity mod 8

A. Korzeniewski (geometric theorem, 2005)

Theorem Let $F^{4m} \longrightarrow E^{4n+4m} \longrightarrow B^{4n}$ be a fibre bundle such that the action of $\pi_1(B)$ on $(H_{2m}(F;\mathbb{Z})/torsion) \otimes \mathbb{Z}_2$ is trivial then

$$\sigma(E) \equiv \sigma(F)\sigma(B) \pmod 8$$

A. Korzeniewski (algebraic theorem, 2005)

Theorem Let (C, ϕ) be a $4n$ dimensional visible symmetric complex over $\mathbb{Z}[\pi_1(B)]$ and let (A, α, U) be a $\mathbb{Z}_2$-trivial $(\mathbb{Z}, 2m)$-symmetric representation. Then

$$\sigma((A, \alpha, U)\widetilde{\otimes}(C, \phi)) \equiv \sigma(C, \phi)\sigma(A, \alpha) \pmod 8$$

Project: What happens if the $\mathbb{Z}_2$-triviality condition is not assumed?

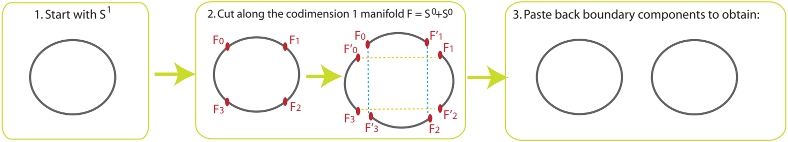
Summary of multiplicativity

		$\pi_1(B)$ acts trivially on $H^*(F,\mathbb{Q})$	$\pi_1(B)$ acts trivially on $H^{2m}(F,\mathbb{Z})/torsion \otimes \mathbb{Z}_2$	No assumption
Double covers	Manifolds	Multiplicative $\sigma(E)=2\sigma(B)$	Multiplicative $\sigma(E)=2\sigma(B)$	Multiplicative $\sigma(E)=2\sigma(B)$
	Geom. Poincaré cx	Multiplicative	Multiplicative mod 8[1] $2\sigma(W)-\sigma(\overline{W})=8[s(W)]$	Multiplicative mod 8
Fibrations in general	Manifolds	Multiplicative	Multiplicative mod 8	Not mult. in general (Atiyah & Kodaira)
	Geom. Poincaré cx	Multiplicative	Multiplicative mod 8	Not mult. in general Mult. mod 4

[1] The total surgery obstruction $s(W) \in \mathbb{S}_{4k}(W)$ has image the codimension 1 splitting obstruction $[s(W)]$, which is given by the Browder-Livesay invariant.

Cut and paste operations and the *SK*-groups

Cut and paste operations on a manifold M are realized as follows: Cut a closed n-dimensional smooth manifold M along a codimension 1 manifold F which has trivial normal bundle. After performing this cut we obtain a manifold with two boundary components, each of them a copy of F. Pasting back these boundary components by a diffeomorphism $h : F \to F$, results in a new manifold $M(F, h)$.



The set of equivalence classes of oriented manifolds in a space X modulo the relation created by cutting and pasting gives rise to the definition of *SK*-groups.

The signature is a cut and paste invariant.

Jänich (1968)

Let $A = M_1 \cup_h M_2$ and $B = M_1 \cup_g M_2$ be two closed n-dimensional manifolds, and $h, g : \partial M_1 \to \partial M_2$ be orientation reversing diffeomorphisms. By the Novikov additivity of the signature:

$$\sigma(A) = \sigma(M_1 \cup_h M_2) = \sigma(M_1) + \sigma(M_2) = \sigma(M_1 \cup_g M_2) = \sigma(B).$$

Hence the signature is a cut and paste invariant.

SK groups and the multiplicativity of the signature I.

Let $F_n(X) \subseteq \Omega_n(X)$ be the subgroup of the bordism classes of closed n-dimensional manifolds which fibre over S^1, then the cut and paste bordism groups are defined **geometrically** as $\overline{SK}_n(X) \cong \Omega_n(X)/F_n(X)$

Neumann (1975)

Theorem (Neumann) If $F^m \to E^{4k} \to B^n$ is a fibration with $\sigma(B) = 0$ and $\sigma(E) \neq 0$, so that $\sigma(E) \neq \sigma(F)\sigma(B)$, then $[B, f : B \to BG]$ generates a free $\overline{SK}_*(BG)$-module.

The asymmetric signature of a mapping torus is zero:

$$\sigma Asy(T(h)) = 0 \in LAsy^*(\mathbb{Z}[\pi_1(T(h)])$$

Ranicki (1998)

$$\overline{SK}(X) \cong \mathrm{Im}(\sigma Asy : \Omega_n(X) \longrightarrow LAsy^n(\mathbb{Z}[\pi_1(X)])$$

SK groups and the multiplicativity of the signature II.

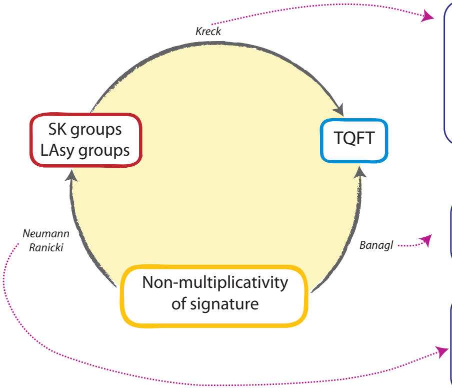
Neumann's theorem can be proved **algebraically** by using the transfer map in asymmetric *L*-theory:

$$\begin{array}{ccccccc} LAsy^n(\mathbb{Z}[\pi_1(B)]) & \xrightarrow{p^!} & LAsy^{4k}(\mathbb{Z}[\pi_1(E)]) & \longrightarrow & LAsy^{4k}(\mathbb{Z}) & \longrightarrow & L^{4k}(\mathbb{Z}) = \mathbb{Z} \\ \sigma Asy(B) & \longmapsto & \sigma Asy(E) & & \longmapsto & & \sigma(E). \end{array}$$

Note that if $\sigma(E) \neq 0$ then $\sigma Asy(B)$ has infinite order in $LAsy^n(\mathbb{Z}[\pi_1(B)])$.

Consequently $0 \neq \sigma Asy(B) \in \text{Im}(\sigma Asy : \Omega_n(B) \longrightarrow LAsy^n(\mathbb{Z}[\pi_1(B)])$.

Relations diagram



From a topological point of view, a field theory gives a way to compute an invariant by cutting a manifold into simple pieces and making the function effectively computable. Which invariants of closed manifolds are partition functions of the field theory? No complete answer is known to this quetion, but SK invariants are such partition functions.

Very recently Markus Banagl has described how non-multiplicative fibre bundles give rise to certain TQFTs.

This relationship was described geometrically by Neumann and algebraically in terms of asymmetric L-theory by Ranicki

Thank you!