An Introduction to Game Theory

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Game Theory Nobel Prize winners

- Lloyd Shapley 2012
- Alvin Roth 2012
- Roger B. Myerson 2007
- Leonid Hurwicz 2007
- Eric S. Maskin 2007
- Robert J. Aumann 2005
- Thomas C. Schelling 2005
- William Vickrey 1996
- Robert E. Lucas Jr. 1995
- John C. Harsanyi 1994
- John F. Nash Jr. 1994
- Reinhard Selten 1994
- Kenneth J. Arrow 1972
- Paul A. Samuelson 1970

Zero Sum Games–John von Neumann

The rules, the game:

I made a game effort to argue but two things were against me: the umpires and the rules.–Leo Durocher

Zero Sum Games

In a simplified analysis of a football game suppose that the offense can only choose a pass or run, and the defense can choose only to defend a pass or run. Here is the matrix in which the payoffs are the average yards gained:

	Defense	
Offense	Run	Pass
Run	1	8
Pass	10	0

The offense's goal is to maximize the average yards gained per play. The defense wants to minimize it.

In the immortal words of mothers everywhere:

You can't always get what you want–Rolling Stones.

Zero Sum Games

Pure saddle point row i^*, column j^*:

$$\boxed{a_{ij^*} \le a_{i^*j^*} \le a_{i^*j}, \text{ for all } i, j}$$

	Defense	
Offense	Run	Pass
Run	1	8
Pass	10	0

No PURE strategies as a **saddle point**: **Largest in row and smallest in column–simultaneously.**
Makes sense–otherwise it would be optimal to always choose same play.

Mixed Strategies-Von Neumann's idea

Player I chooses a probability of playing each row.

$$X = (x_1, x_2, \dots, x_n).\ \ x_i \in [0,1], \sum_{i=1}^{n} x_i = 1$$

Player II chooses a probability of playing each column.

$$Y = (y_1, y_2, \dots, y_m),\ \ y_j \in [0,1], \sum_{j=1}^{m} y_j = 1.$$

Expected Payoff to Player I: $E(X,Y) = \sum_{i,j} a_{ij} x_i y_j = XAY^T$.
Mixed Saddle Point X^*, Y^*, $v =$ value of the game.

$$\boxed{E(X, Y^*) \le E(X^*, Y^*) = v \le E(X^*, Y),\ \text{for all } X, Y}$$

	Defense	
Offense	Run	Pass
Run	1	8
Pass	10	0

$$X^* = (.588, .412), Y^* = (.47, .53), value = 4.70.$$

is a mixed saddle point and 4.7 is the value of the game.

What if the Bears hired Peyton Manning?

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Offense	Run	Pass
Run	1	8
Pass	14	2

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The percentage of time to defend against the pass goes up.

$$(.47, .53) \to Y^* = (.32, .68), 4.70 \to \mathit{value} = 5.79.$$

.

Everyone has a plan until they get hit.–Mike Tyson, Heavyweight Boxing Champ 1986-1990.

How should drug runners avoid the cops?

Drug runners can use three possible methods for running drugs through Florida: small plane, main highway, or backroads.
The cops can only patrol one of these methods at a time.

- Profit:
  - Use Highway \$100,000
  - Use backroads \$80,000.
  - Fly \$150,000.
- Penalties:
  - Highway– \$90,000
  - Backroads–\$70,000
  - Plane–\$130,000 if by small plane.

Data: Chance of getting Caught if cops patrolling that method: (1) Highway–40%. (2) Backroads–30%. (3) Plane–6-%.

Solution

The game matrix becomes

	Cops		
Runners	Plane	Highway	Road
Plane	-18	150	150
Highway	100	24	100
Road	80	80	35

For example, if drug runner plays Highway and cops patrol Highway the drug runner's expected payoff is $(-90)(0.4) + (100)(0.6) = 24$. The saddle point is

$$X^* = (0.14, 0.32, 0.54) \quad Y^* = (0.46, 0.36, 0.17), v = 72.25.$$

The drug runners should use the back roads more than half the time, but the cops should patrol the back roads only about 17% of the time.

Non Zero Sum Games–John Nash

I returned, and saw under the sun, that the race is not to the swift, nor the battle to the strong, . . . ; but time and chance happeneth to them all–Ecclesaistes 9:11

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The race is not always to the swift nor the battle to the strong, but thats the way to bet.
Damon Runyon, More than Somewhat

Non Zero Sum Games

Now each player gets their own payoff matrix which we write as one bimatrix:

Example: Two airlines A,B serve the route ORD to LAX. Naturally, they are in competition for passengers who make their decision based on airfares alone. Lower fares attract more passengers and increases the load factor (the number of bodies in seats). Suppose the bimatrix is given as follows where each airline can choose to set the fare at Low or High :

A/B	Low	High
Low	(-50,-10)	(175,-20)
High	(-100,200)	(100,100)

Each player wants their payoff as large as possible.

Nash Equilibrium

Given payoff functions $u_1(x,y), u_2(x,y)$, a Nash equilibrium is a point (x^*, y^*) so that

$$u_1(x^*, y^*) \geq u_1(x, y^*), \text{ for all other strategies } x$$

and

$$u_2(x^*, y^*) \geq u_2(x^*, y), \text{ for all other strategies } y$$

Another way to say this is

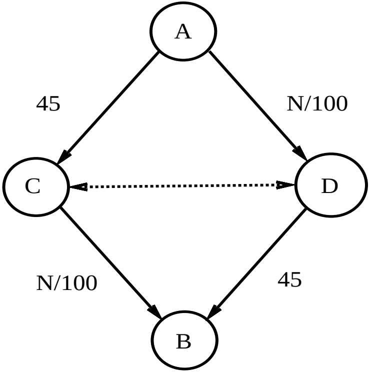
$$u_1(x^*, y^*) = \max_x u_1(x, y^*), \quad u_2(x^*, y^*) = \max_y u_2(x^*, y).$$

A/B	Low	High
Low	**$(-50, -10)$**	(**175**,-20)
High	(-100,**200**)	(100,100)

An example of a **Prisoner's Dilemma game**.

Braess's Paradox

Figure: Braess Paradox



N commuters want to travel from A to B. The travel times:

- $A \to D$ and $C \to B$ is $N/100$
- $A \to C$ and $D \to B$ is 45

Each player wants to minimize her own travel time. Total travel time for each commuter $A \to D \to B$ is $N/100 + 45$, and the total travel time $A \to C \to B$ is also $N/100 + 45$.

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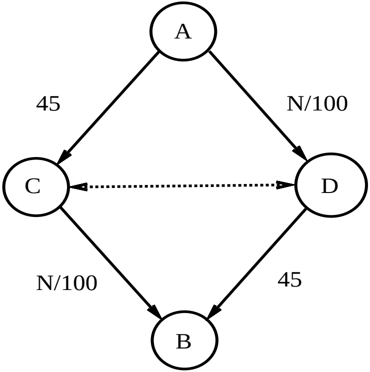

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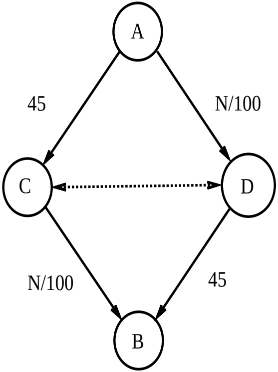
Nash equilibrium : $N/2$ players take each route.

Braess continued



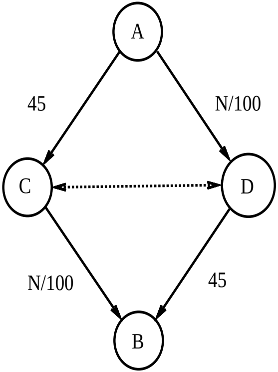
$N = 4000$ commuters want to travel from A to B. The travel times:

- $A \to D$ and $C \to B$ is $N/100$
- $A \to C$ and $D \to B$ is 45

Without zip road each uses NE: travel time for each commuter $A \to D \to B$ is $2000/100 + 45 = 65$, and travel time $A \to C \to B$ is also $2000/100 + 45 = 65$.

If anyone deviates his travel time is $\frac{2001}{100} + 45 = 65.01$

Braess's Paradox-end



With zip road: If $N = 4000$ then all commuters would pick $A \to D$ taking $N/100 = 40$; then take the zip road to C and travel from $C \to B$.**Total commute time would be** 80.
If they skip the zip road, travel time would be $40 + 45 = 85 > 80$ so they will take the zip road. **Zip road makes things worse!**

Cooperative Games–Lloyd Shapley, John von Neumann

We must all hang together, or assuredly we will all hang separately. Benjamin Franklin, at the signing of the Declaration of Independence

Cooperative Games

$N = \{1, 2, 3, \ldots, n\}$ players who seek to join a coalition so they all can do better.
$S \subset N$ is a **coalition**.
$v : 2^N \to \mathbb{R}$ is a **characteristic function** if $v(\emptyset) = 0$ and is superadditive:

$$v(S \cup T) \geq v(S) + v(T), \forall S, T \subset N, S \cap T = \emptyset.$$

$v(N) =$ rewards of **grand coalition** in which everyone cooperates.

Central question: How are the rewards $v(N)$ allocated to each of the N players?

Imputations-Allocations

A vector $\vec{x} = (x_1, x_2, \ldots, x_n)$ is an **imputation** if

- $x_i \geq v(i), i = 1, 2, \ldots, n$–**individual rationality**
- $\sum_{i=1}^{n} x_i = v(N)$–**group rationality.**

How to find $\vec{x}$ as a **fair** allocation?

Nucleolus and Shapley Value

Two concepts of solution:

- **Nucleolus** (Von Neumann and Morgenstern)– the fair allocation is chosen so as to minimize the maximum dissatisfaction over all possible allocations and all possible coalitions. (Core, Least Core)
- **Shapley Value**: A fair allocation to player i is the mean of the worth of player i to any coalition.

Shapley Value:

$$\boxed{x_i = \sum_{S \in \Pi^i} \frac{(|S|-1)!(n-|S|)!}{n!}\left(v(S) - v(S-i)\right)}$$

Investment game

Companies can often get a better cash return if they invest larger amounts. There are 3 companies who may cooperate to invest money in a venture that pays a rate of return as follows:

Invested Amount	Rate of Return
0-1,000,000	4%
1,000,000-3,000,000	5%
>3,000,000	5.5%

Suppose Company 1 will invest $1,800,000, Company 2, $900,000, and Company 3, $400,000. If they all invest they NET $170500. How should this interest be split among the three companies?

Solution of Investment game

Invested Amount	Rate of Return	Company Invests
0-1,000,000	4%	(1)1800000
1,000,000-3,000,000	5%	(2) 900000
>3,000,000	5.5%	(3) 400000

The characteristic function is interest earned on the investment:

$$v(1) = 90,000,\ v(2) = 36,000,\ v(3) = 16,000$$
$$v(12) = 135,000,\ v(13) = 110,000\ v(23) = 65,000$$
$$v(123) = 170,500$$

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$$v(123) = 170,500$$

The nucleolus is $x_1 = 97750, x_2 = 47000, x_3 = 25750.$

$$\text{Nucleolus: } x_1 = 97750, x_2 = 47000, x_3 = 25750.$$

Compare: Proportional payment: commonly the fair allocation each player will get the amount of 170500 proportional to the amount they invest.

$$y_1 = \frac{90}{142}170500 = 108063.38, y_2 = \frac{36}{142}170500 = 43225.35,$$

$$y_3 = \frac{16}{142}170500 = 19211.27.$$

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Does not take into account that player 3 is a very important investor. It is her money that pushes the grand coalition into the 5.5% rate of return. Without player 3 the most they could get is 5%. Consequently player 3 has to be compensated for this power. The proportional allocation doesn't do that.

Shapley Solution

$v(1) = 90, \; v(2) = 36,$
$v(3) = 16$
$v(12) = 135$
$v(13) = 110,$
$v(23) = 65$
$v(123) = 170.5$

Order	1	2	3
123	90	45	35.5
132	90	60.5	20
213	99	36	35.5
231	105.5	36	29
312	94	60.5	16
321	105.5	49	16
Shapley	97.33	47.83	25.33

Compare

Method	x_1	x_2	x_3
Nucleolus	97750	47000	25750
Proportional	108063	43225	19211
Shapley	97333	47833	25333

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If you are player 1, your lawyer argues for Proportional.

How much power do conservatives have?

Senate 112th Congress has 100 members: 53 are Democrats and 47 are Republicans.

3 types of Democrats and 3 types of Republicans:

- Liberals,
- Moderates,
- Conservatives.

Assume that these types vote as a block.

A resolution requires 60 votes to pass.

- Democrats:
  - Liberals(1) 20 votes
  - Moderates(2) 25 votes
  - Conservatives(3) 8 votes.
- Republicans:
  - Liberals(4) 2 votes
  - Moderates(5) 15 votes
  - Conservatives(6) 30 votes.

Power of Repubs and Democrats

Define the characteristic function $v(S) = \begin{cases} 1, & \text{if } |S| \geq 60; \\ 0, & \text{if } |S| < 60. \end{cases}$

We find the **Shapley-Shubik index** and the total power of the Republicans and Democrats.

A straightforward computation using the Shapley formulas gives

$$x_1 = 21.67\%,\ x_2 = 25\%,\ x_3 = 5\%,\ x_4 = 1.67\%,\ x_5 = 16.67\%,\ x_6 = 30\%.$$

The total Democratic power is $x_1 + x_2 + x_3 = 51.67\%$ and Republican power is $x_4 + x_5 + x_6 = 48.33\%$

Republicans continued

What happens if the Republican Moderate votes becomes 1, while the Republican Conservative votes becomes 44.
The Shapley-Shubik index in this case is

$$x_1 = 16.67\%,\ x_2 = 16.67\%,\ x_3 = 0\%,\ x_4 = 0\%,\ x_5 = 0\%,\ x_6 = 66.67\%.$$

The total Democratic power is $x_1 + x_2 + x_3 = 33.34\%$ and Republican power is $x_4 + x_5 + x_6 = 66.67\%$
The Republicans, a minority in the Senate, have dominant control due to the conservative bloc. The conservatives in the Democratic party, and the moderates and liberals in the Republican party have no power at all.

The End

- **Don't let it end like this. Tell them I said something.—Pancho Villa**
- **In the end everything is a gag.–Charlie Chaplin.**
- **Start every day off with a smile and get it over with.–W.C.Fields**