Errata for Game Theory: An Introduction, by E.N.Barron

Please notify me at ebarron@luc.edu for any errors. I am grateful to all of those mentioned below who notified me of the errors mentioned. The following list of errors is current as of May 3, 2010.

1. Stephen Conwill found the following errors.
   (a) p.8 In the table at the bottom of the page II3 should be the strategy: *If I1, then S; If I2, then S.* The strategy II4 should be: *If I1, then S; If I2, then P.*
   (b) p. 9 line 5 from the top "pass as well" should be *spin*.
2. Dinesh Ayyappan found the following error.
   (a) p.11 line 5 from top, the word "largest" should be replaced by the word "smallest".
3. p. 12 Lemma 1.1.3, second line of proof should be
$$v^+ = \min_j \max_i a_{i,j} \le \max_i a_{i,j^*} \le a_{i^*,j^*} \le \min_j a_{i^*,j} \le \max_i \min_j a_{i,j} = v^-.$$
   p. 12 in proof of Lemma 1.1.3,"Let i^* be such that . . . $j = 1, 2, m$. Should be: "Let j^* be such that $v^+ = \max_i a_{i,j^*}$ and i^* such that $v^- = \min_j a_{i^*,j}$. Then
$$a_{i^*,j} \ge v^- = v^+ \ge a_{i,j^*}, \text{ for any } i = 1, 2, , n, j = 1, 2, , m.$$
4. p. 16, line 6, $v^+ = \min_{x\in C} \max_{y\in D} f(x, y)$, and $v^- = \max_{y\in D} \min_{x\in C} f(x, y)$, should be
$$v^+ = \min_{y\in D} \max_{x\in C} f(x, y), \text{ and } v^- = \max_{x\in C} \min_{y\in D} f(x, y).$$
5. p. 22, The last line of the third paragraph "These probability vectors are called mixed strategies, and will turn out to be the class correct class of strategies for each of the players." should be "These probability vectors are called mixed strategies, and will turn out to be the correct class of strategies for each of the players."
6. p. 31, last line, remove ")."
7. p. 34, line 3 from top, "...property 3,..." should be "...properties 3 and 5..."
8. The following errors were found by Yan Jin .
   (a) p. 43, line 12 from bottom, $E(4, Y) = -5y + 6(1 - y)$ should be $E(4, Y) = 7y - 8(1 - y)$.
   (b) p. 44, line 1 from top, $E(1, X)$ should be corrected as $E(X, 1)$. Line 2 from top, $E(4, X)$ should be $E(X, 2)$, and $(x = 5/6, 1/3)$ should be corrected as $(x = \frac{5}{6}, v = \frac{1}{3})$.
9. p. 47, Problem 1.29, part (a) should have $\min_j E(X, j) = -\frac{42}{9}$.
10. p. 55, Quotation added
11. The following errors were also found by Yan Jin
    (a) p. 68, the second line of the proof of Theorem 2.3.1 should read
    $E(X, X) = XAX^T = -XA^TX^T = -(XA^TX^T)^T = -XAX^T = -E(X, X)$.
    In other words, the third A should be A^T.
    (b) p. 69, the third line from the bottom, $(a\lambda, -b\lambda, c\lambda)$ should be $(c\lambda, -b\lambda, a\lambda)$.

12. p. 90 problem 2.34, Remove part (a) and (b) and change hint to:

    Hint: Player I has 4 strategies, e.g., If ace, bet 2; If jack, bet 2. Player II also has 4 strategies, e.g., If I bets 4, then Fold; If I bets 2, then Call. Player I's strategies are then, $(2,2),(2,4),(4,2),(4,4)$, where the first number is the amount to bet if an ace. Player II's strategies are $(F,C),(C,C),(F,F),(C,F)$, where the first letter is for a bet of 4.

13. p. 111, line 7 from top E_2 should be E_{II}.

14. Joe Condon found the following errors.

    (a) p.117 line 1, $v(B^T)) = \frac{1}{4}$ should be $v(B^T) = \frac{3}{4}$. In line 8 from the top, $X^{B^T} = (\frac{1}{4}, \frac{3}{4})$ should be $X^{B^T} = (\frac{3}{4}, \frac{1}{4})$. Line 9 from the top "value of $\frac{1}{4}$" should be "value of $\frac{3}{4}$."

    (b) p.123, The case $R < 0$ should have the possible solutions

$$\text{if } y = 0 \implies 1 \geq x \geq \frac{r}{R},$$
$$\text{if } 0 < y < 1 \implies x = \frac{r}{R},$$
$$\text{if } y = 1 \implies 0 \leq x \leq \frac{r}{R}.$$

    In addition, in line 2 from the bottom $R < 0$ should be $R > 0$, the figure 3.2 should have $R > 0$, and line 2 above the figure should have $R > 0$.

15. p. 125, line 9 from bottom, $E(1,Y)$ should be $E_{\text{I}}(1,Y)$.

16. p. 137, Matrix B, first row should be $[0,1,-1]$, not $[0,2,1]$. Then $value(B^T) = \frac{2}{3}, X_B = (\frac{1}{3}, \frac{2}{3}, 0), Y_B = (\frac{2}{3}, \frac{1}{3})$. Fu-Te Hsu kindly pointed this out.

17. p. 145, line 5 from bottom, Y^*T should be Y^{*T}.

18. p. 154, problem 3.23 has the answer fixed on p. 393: should have $f(x,y,p,q) = 7x+7y-6xy-6-p-q$, and $2 - x \leq q$ should be $2x - 1 \leq q$. Problem 3.25 should read "Find as many as you can by...."

19. p. 176, line 3 from top, replace Q by q.

20. p. 181, line 3 from bottom, should have $u_1(q_1^*, q_2^*) = \frac{(\Gamma - 2c_1 + c_2)^2}{8}$.

21. p. 182, line 2 from bottom, replace γ by Γ. Line 4 from the top should be $u_1(q_1^*, q_2^*) = \dfrac{(\Gamma - 2c_1 + c_2)^2}{8}$.

    Line 8 from the bottom should be $u_1(q_1^*, q_2^*) = \dfrac{(\Gamma - c)^2}{8}$.

22. p. 184, line 10 from bottom, q_1 should be q_1^0 in three spots.

23. p. 185, (i) Problem 4.3 should be changed to "Compare profits for firm 1 in the model with uncertain costs and the standard Cournot model. Assume $\Gamma = 15; c_1 = 4; c^+ = 5; c^- = 1$ and $p = 0.5$."

    (ii) Problem 4.6 : Should be: Suppose that two firms have constant unit costs $c_1 = 2, c_2 = 1$, and $\Gamma = 19$ in the Stackelberg model.

24. p. 194, Problem 4.17, line 7 from bottom, "... each variable separately ..." should be "... the variable they control ...."

25. p. 221, Example 5.1(4): "but will take \$1 million ...," should be "but will take \$100 million ...."

26. p. 224, In matrix for player 3 versus coalition players 12, entry for A versus AB should be $+2$.

27. p. 225, line 9 from bottom, "...and the assistance of player 1 doesn't help since $v(13) = -1$." should be "but the assistance of player 1 does help since $v(13) = 1$."

28. p. 227, Professor Kevin Easley kindly pointed out the following error: The first sentence of Definition 5.1.2 should be replaced by "Let x_i be a real number for each $i = 1, 2, \ldots, n$"

29. p. 227, line 12 from bottom, $\sum_i v(i) \geq \sum_i x_i$ should be $\sum_i x_i \geq \sum_i v(i)$

30. p. 240, Problem 5.2(b) solution $\frac{38}{5}$ should be $\frac{22}{5}$.

31. p. 241, Problem 5.10: $x - 2$ should be x_2.

32. p. 243, first line of second paragraph, "... give away to the whomever ..." should be "... give away to whomever ..."

33. p. 244, line 2 from top, x should be $\vec{x}$ in two places.

34. p. 245, Remark 3, $e(S, \vec{x})$ missing in two spots. Last line should have $\vec{x}$ and $\vec{x}^*$.

35. p. 246, `x1+x2+x3=\frac{5}{2}` should be $x1 + x2 + x3 = 5/2$.

36. p. 253, line 15 from top, `\frac {9}{10}` should be 9/10.

37. p. 259, line 6 from bottom, `\frac {11}{12}` should be 11/12 in three places.

38. p. 260, line 3 from bottom, c_{13} should be 18, not 10.

39. p. 265, Last paragraph before Example 5.14 should have a last sentence: At the end of this chapter you can find the Maple code to find the Shapley value.

40. p. 275, line 4 from top, n should be 4.

41. p. 306, The Maple code for the calculation of the Shapley value is added.

42. p. 388, Problem 2.21 should have solution $X^* = (0, \frac{5}{11}, \frac{5}{11}, 0, \frac{1}{11}) = Y^*$.

43. p. 393, Problem 3.24, $Y_1 = (\frac{5}{13}, \frac{5}{13}, \frac{2}{13})$ should be $Y_1 = (\frac{6}{13}, \frac{5}{13}, \frac{2}{13})$.

44. p. 394, Problem 3.27(c), $Y_1 = (\frac{5}{13}, \frac{5}{13}, \frac{2}{13})$ should be $Y_1 = (\frac{6}{13}, \frac{5}{13}, \frac{2}{13})$.

45. p. 395, Problem 4.3 should have the answer "Profit for firm 1 is 10, compared with 16 or 7.11 if $c_2 = 5$ or $c_2 = 1$, resp.."

46. p. 400 Problem 5.9 "since $-x - 1$ ..." should be "... since $-x_1$ ...." . Problem 5.13 should have $16 - x_1 - x_2$, not $16 - x_1 - x - 2$.

47. p. 401, Problem 5.19 solution in (b) should have $x_4 = \frac{3}{2}$, not 32.

48. p. 402, Problem 5.20, line 1 "The characteristic function for the savings game is ...."

49. p. 404-405, Problem 6.5 should have solutions (b) and (c) switched.

50. p. 409, Reference [7], should be *Game Theory: Mathematical Models of Conflict*

Professor Kevin Easley and his Game Theory class found the following errors and clarifications. I am grateful to Professor Easley and all the members of his class for their careful reading.

1. p. 47 Exercise 1.28 modified to

Show that if (X^*, Y^*) and (X^0, Y^0) are both saddle points for the game with matrix A, then so is (X^*, Y^0) and (X^0, Y^*); so is, (X_λ, Y_β) where $X_\lambda = \lambda X^* + (1-\lambda)X^0, Y_\beta = \beta Y^* + (1-\beta)Y^0$ and λ, β any numbers in $[0,1]$.

2. p. 227 Change Definition 5.1.2 to

   Let x_i be a real number for each $i = 1, 2, \ldots, n$, with $\sum_i x_i \le v(N)$. A vector $\vec{x} = (x_1, \ldots, x_n)$ is an **imputation** if ...

3. p. 227 Remark 3. Change to:

   Group rationality means any increase of reward to a player must be matched by a decrease in reward for one or more other players. Why is group rationality reasonable? Well, we know that $v(N) \ge \sum_i x_i \ge \sum_i v(i)$, just by definition. If in fact $\sum_i x_i < v(N)$, then each player could actually receive a bigger share than simply x_i; in fact, one possibility is an additional amount $(v(N) - \sum_i x_i)/n$. This says that the allocation x_i would be rejected by each player, so it must be true that $\sum_i x_i = v(N)$ for any reasonable allocation.

4. p. 228, Line 7. should read "any two n-person cooperative games."

5. p. 228, Lemma 5.1.3, line 2. "... normalization with characteristic function v" should be "... normalization with characteristic function v' " with a prime on the v.

6. p. 230, Definition 5.1.5, the excess function is defined for imputations $\vec{x} \in X$ and not just $\vec{x} \in R$.

7. p. 240 Exercise 5.3, should be "Given the characteristic function $v(i) = 0, i = 1, 2, 3, 4$ and ...".

8. p. 241, Exercise 5.9, should be "... and if $x \in C(0)$, then ...".

9. p. 244, Last line preceding Lemma 5.1.10: text should read " ... or there is more than one allocation in $C(0)$."

10. p. 246, Line 22, The defining conditions for $C(\varepsilon)$ should also have $S \subsetneq N$ and $S \neq \emptyset$.

11. p. 249, Three lines from bottom, Should also include the condition $S \neq \emptyset$.

12. p. 279, line 2, Should be "... for some ...."

13. p. 280 Just before Definition 5.4.1. "The entire triangle in Figure 5.9 ..." should be "The entire 4-sided region ...."

14. p. 291, Example 5.26, change "utility" to "payoff."

15. p. 291, Example 5.26. Line 7, The comma after $ln(y+1)$ is misplaced to the right.

16. p. 296, Just below line 2 from top should have the sentence: "It is proved in [7] that this line *must* pass through the optimal threat security point."

17. p. 296, Line 4 from bottom, $m_p\, u^t + v^t$ should be $-m_p\, u^t - v^t = \frac{3}{8}\, u^t - v^t$. Line 2 from bottom also $m_p\, u^t + v^t$ should be $-m_p\, u^t - v^t$.

18. p. 380, Footnote, The name "Rockafeller" is misspelled and should be "Rockefeller"

tree, which is nothing more than a picture of what happens at each stage of the game where a decision has to be made.

The numbers at the end of the branches are the payoffs to player I. The number $\frac{1}{2}$, for example, means that the net gain to player I is \$500 because player II had to pay \$1000 for the ability to pass and they split the pot in this case. The circled nodes are spots at which the next node is decided by chance. You could even consider Nature as another player. We analyze the game by first converting the tree to a game matrix which, in this example becomes

I/II	II1	II2	II3	II4
I1	$\frac{1}{4}$	$\frac{1}{4}$	$-\frac{1}{36}$	$-\frac{1}{36}$
I2	$-\frac{3}{2}$	0	$-\frac{3}{2}$	0

To see how the numbers in the matrix are obtained, we first need to know what the pure strategies are for each player. For player I, this is easy because she makes only one choice and that is pass (I2) or spin (I1). For player II, II1 is the strategy; if I passes, then spin, but if I spins and survives, then pass. So, the expected payoff[2] to I is

$$\text{I1 against II1}: \frac{5}{6}\left(\frac{1}{2}\right) + \frac{1}{6}(-1) = \frac{1}{4}, \quad \text{and}$$
$$\text{I2 against II1}: \frac{5}{6}(-2) + \frac{1}{6}(1) = -\frac{3}{2}.$$

Strategy II3 says the following: If I spins and survives, then spin, but if I passes, then spin and fire. The expected payoff to I is

$$\text{I1 against II3}: \frac{5}{6}\left(\frac{5}{6}(0) + \frac{1}{6}(1)\right) + \frac{1}{6}(-1) = -\frac{1}{36}, \quad \text{and}$$
$$\text{I2 against II3}: \frac{5}{6}(-2) + \frac{1}{6}(1) = -\frac{3}{2}.$$

The remaining entries are left for the reader. The pure strategies for player II are summarized in the following table.

II1	If I2, then S;	If I1, then P.
II2	If I2, then P;	If I1, then P.
II3	If I1, then S;	If I2, then S.
II4	If I1, then S;	If I2, then P.

[2]This uses the fact that if X is a random variable taking values $x_1, x_2, \ldots, x_n$ with probabilities $p_1, p_2, \ldots, p_n$, respectively, then $EX = \sum_{i=1}^{n} x_i p_i$. In I1 against II1, X is $\frac{1}{2}$ with probability $\frac{5}{6}$ and -1 with probability $\frac{1}{6}$. See the appendix for more.

This is actually a simple game to analyze because we see that player II will never play II1, II2, or II4 because there is always a strategy for player II in which II can do better. This is strategy II3, which stipulates that if I spins and survives the shot, then II should spin, while if I passes, then II should spin and shoot. If I passes, II gets $\frac{1}{36}$ and I loses $-\frac{1}{36}$. If I spins and shoots, then II gets $\frac{3}{2}$ and I loses $-\frac{3}{2}$. The larger of these two numbers is $-\frac{1}{36}$, and so player I should always spin and shoot. Consequently, player II will also spin and shoot.

The dotted line in Figure 1.3 indicates the optimal strategies. The key to these strategies is that no significant value is placed on surviving.

Remark. Extensive form games can take into account information that is available to a player at each decision node. This is an important generalization. Extensive form games are a topic in sequential decision theory, a second course in game theory.

Finally, we present an example in which is it clear that randomization of strategies must be included as an essential element of games.

■ EXAMPLE 1.6

Evens or Odds. In this game, each player decides to show one, two, or three fingers. If the total number of fingers shown is even, player I wins $+1$ and player II loses -1. If the total number of fingers is odd, player I loses -1, and player II wins $+1$. The strategies in this game are simple: deciding how many fingers to show. We may represent the payoff matrix as follows:

Evens		Odds	
I/II	1	2	3
1	1	-1	1
2	-1	1	-1
3	1	-1	1

The row player here and throughout this book will always want to maximize his payoff, while the column player wants to **minimize** the payoff to the row player, so that her own payoff is maximized (because it is a zero sum game). The rows are called the **pure strategies** for player I, and the columns are called the **pure strategies** for player II.

The following question arises: How should each player decide what number of fingers to show? If the row player **always** chooses the same row, say, one finger, then player II can **always** win by showing two fingers. No one would be stupid enough to play like that. So what do we do? In contrast to 2×2 Nim or Russian roulette, there is no obvious strategy that will always guarantee a win for either player.

Even in this simple game we have discovered a problem. If a player always plays the same strategy, the opposing player can win the game. It seems

For each row, find the minimum payoff in each column and write it in a new additional last column. Then the lower value is the largest number in that last column, that is, the maximum over rows of the minimum over columns. Similarly, in each column find the maximum of the payoffs (written in the last row). The upper value is the smallest of those numbers in the last row.

$$\begin{array}{cccc|l}
a_{11} & a_{12} & \cdots & a_{1m} & \longrightarrow \min_j a_{1j} \\
a_{21} & a_{22} & \cdots & a_{2m} & \longrightarrow \min_j a_{2j} \\
\vdots & \vdots & \cdots & \vdots & \\
a_{n1} & a_{n2} & \cdots & a_{nm} & \longrightarrow \min_j a_{nj} \\
\hline
\downarrow & \downarrow & \cdots & \downarrow & \\
\max_i a_{i1} & \max_i a_{i2} & \cdots & \max_i a_{im} & \quad v^- \;= \text{largest min} \\
 & & & & \quad v^+ \;= \text{smallest max}
\end{array}$$

Here is the precise definition.

Definition 1.1.1 *A matrix game with matrix $A_{n\times m} = (a_{ij})$ has the lower value*

$$v^- \equiv \max_{i=1,\ldots,n} \min_{j=1,\ldots,m} a_{ij}.$$

and the upper value

$$v^+ \equiv \min_{j=1,\ldots,m} \max_{i=1,\ldots,n} a_{ij},$$

The lower value v^- is the smallest amount that player I is guaranteed to receive (v^- is player I's gain floor), and the upper value v^+ is the guaranteed greatest amount that player II can lose (v^+ is player II's loss ceiling). The **game has a value** *if $v^- = v^+$, and we write it as $v = v(A) = v^+ = v^-$. This means that the smallest max and the largest min must be equal and the row and column i^*, j^* giving the payoffs $a_{i^*,j^*} = v^+ = v^-$ are* **optimal***, or a* **saddle point in pure strategies***.*

One way to look at the value of a game is as a handicap. This means that if the value v is positive, player I should pay player II the amount v in order to make it a **fair game**, with $v = 0$. If $v < 0$, then player II should pay player I the amount $-v$ in order to even things out for player I before the game begins.

■ **EXAMPLE 1.7**

Let's work this out using 2×2 Nim.

$$\begin{array}{rrrrrr|cc}
1 & 1 & -1 & 1 & 1 & -1 & \longrightarrow & \min = -1 \\
-1 & 1 & -1 & -1 & 1 & -1 & \longrightarrow & \min = -1 \\
-1 & -1 & -1 & 1 & 1 & 1 & \longrightarrow & \min = -1 \\
\hline
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & & \boxed{v^- = -1} \\
\max = 1 & \max = 1 & \max = -1 & \max = 1 & \max = 1 & \max = 1 & \boxed{v^+ = -1} &
\end{array}$$

We see that $v^- =$ largest min $= -1$ and $v^+ =$ smallest max $= -1$. This says that $v^+ = v^- = -1$, and so 2×2 Nim has $v = -1$. The optimal strategies are located as the (row,column) where the smallest max is -1 and the largest min is also -1. This occurs at any row for player I, but player II must play column 3, so $i^* = 1, 2, 3,\ j^* = 3$. The optimal strategies are **not** at **any** row column combination giving -1 as the payoff. For instance, if II plays column 1, then II will play row 1 and receive $+1$. Column 1 is not part of an optimal strategy.

We have mentioned that the most that I can be guaranteed to win should be less than (or equal to) the most that II can be guaranteed to lose, (i.e., $v^- \le v^+$), Here is a quick verification of this fact.

For any column j we know that for any fixed row i, $\min_j a_{ij} \le a_{ij}$, and so taking the max of both sides over rows, we obtain

$$v^- = \max_i \min_j a_{ij} \le \max_i a_{ij}.$$

This is true for any column $j = 1, \dots, m$. The left side is just a number (i.e., v^-) independent of i as well as j, and it is smaller than the right side for any j. But this means that $v^- \le \min_j \max_i a_{ij} = v^+$, and we are done.

Now here is a precise definition of a (pure) saddle point involving only the payoffs, which basically tells the players what to do in order to obtain the value of the game when $v^+ = v^-$.

Definition 1.1.2 *We call a particular row i^* and column j^* a* **saddle point in pure strategies** *of the game if*

$$a_{ij^*} \le a_{i^*j^*} \le a_{i^*j}, \text{ for all rows } i = 1, \dots, n \text{ and columns } j = 1, \dots, m. \tag{1.1.1}$$

Lemma 1.1.3 *A game will have a saddle point in pure strategies if and only if*

$$v^- = \max_i \min_j a_{ij} = \min_j \max_i a_{ij} = v^+. \tag{1.1.2}$$

Proof. If (1.1.1) is true, then

$$v^+ = \min_j \max_i a_{i,j} \le \max_i a_{i,j^*} \le a_{i^*,j^*} \le \min_j a_{i^*,j} \le \max_i \min_j a_{i,j} = v^-.$$

But $v^- \le v^+$ always, and so we have equality throughout and $v = v^+ = v^- = a_{i^*,j^*}$.

On the other hand, if $v^+ = v^-$ then

$$\min_j \max_i a_{i,j} = \max_i \min_j a_{i,j}.$$

Let j^* be such that $v^+ = \max_i a_{i,j^*}$ and i^* such that $v^- = \min_j a_{i^*,j}$. Then

$$a_{i^*,j} \ge v^- = v^+ \ge a_{i,j^*}, \text{ for any } i = 1, \dots, n,\ j = 1, \dots, m.$$

Definition 1.2.1 *Let C and D be sets. A function $f : C \times D \to \mathbb{R}$ has at least one saddle point (x^*, y^*) with $x^* \in C$ and $y^* \in D$ if*

$$f(x, y^*) \le f(x^*, y^*) \le f(x^*, y) \text{ for all } x \in C, y \in D.$$

Once again we could define the upper and lower values for the game defined using the function f, called a **continuous game**, by

$$v^+ = \min_{y \in D} \max_{x \in C} f(x, y), \quad \text{and} \quad v^- = \max_{x \in C} \min_{y \in D} f(x, y).$$

You can check as before that $v^- \le v^+$. If it turns out that $v^+ = v^-$ we say, as usual, that the **game has a value** $v = v^+ = v^-$. The next theorem, the most important in game theory and extremely useful in many branches of mathematics is called the **von Neumann minimax theorem**. It gives conditions on $f, C,$ and D so that the associated game has a value $v = v^+ = v^-$. It will be used to determine what we need to do in matrix games in order to get a value.

In order to state the theorem we need to introduce some definitions.

Definition 1.2.2 *A set $C \subset \mathbb{R}^n$ is* **convex** *if for any two points $a, b \in C$ and all scalars $\lambda \in [0, 1]$, the line segment connecting a and b is also in C, i.e., for all $a, b \in C$, $\lambda a + (1 - \lambda)b \in C, \forall\, 0 \le \lambda \le 1$.*

C is closed if it contains all limit points of sequences in C; C is bounded if it can be jammed inside a ball for some large enough radius. A closed and bounded subset of Euclidean space is compact.

A function $g : C \to \mathbb{R}$ is convex if

$$g(\lambda a + (1 - \lambda)b) \le \lambda g(a) + (1 - \lambda)g(b)$$

for any $a, b \in C, 0 \le \lambda \le 1$. This says that the line connecting $g(a)$ with $g(b)$, namely $\{\lambda g(a) + (1 - \lambda)g(b) : 0 \le \lambda \le 1\}$, must always lie above the function values $g(\lambda a + (1 - \lambda)b),\ 0 \le \lambda \le 1$.

The function is concave if $g(\lambda a + (1 - \lambda)b) \ge \lambda g(a) + (1 - \lambda)g(b)$ for any $a, b \in C, 0 \le \lambda \le 1$. A function is strictly convex or concave, if the inequalities are strict.

Figure 1.4 compares a convex set and a nonconvex set. Also, recall the common calculus test for twice differentiable functions of one variable. If $g = g(x)$ is a function of one variable and has at least two derivatives, then g is convex if $g'' \ge 0$ and g is concave if $g'' \le 0$.

Now the basic von Neumann minimax theorem.

Theorem 1.2.3 *Let $f : C \times D \to \mathbb{R}$ be a continuous function. Let $C \subset \mathbb{R}^n$ and $D \subset \mathbb{R}^m$ be convex, closed, and bounded. Suppose that $x \mapsto f(x, y)$ is concave and $y \mapsto f(x, y)$ is convex. Then*

$$v^+ = \min_{y \in D} \max_{x \in C} f(x, y) = \max_{x \in C} \min_{y \in D} f(x, y) = v^-.$$

In fact, define $y = \varphi(x)$ as the function so that $f(x, \varphi(x)) = \min_y f(x, y)$. This function is well defined and continuous by the assumptions. Also define the function $x = \psi(y)$ by $f(\psi(y), y) = \max_x f(x, y)$. The new function $g(x) = \psi(\varphi(x))$ is then a continuous function taking points in $[0, 1]$ and resulting in points in $[0, 1]$. There is a theorem, called the **Brouwer fixed-point theorem**, which now guarantees that there is a point $x^* \in [0, 1]$ so that $g(x^*) = x^*$. Set $y^* = \varphi(x^*)$. Verify that (x^*, y^*) satisfies the requirements of a saddle point for f.

1.3 MIXED STRATEGIES

Von Neumann's theorem suggests that if we expect to formulate a game model which will give us a saddle point, in some sense, we need convexity of the sets of strategies, whatever they may be, and convexity-concavity of the payoff function, whatever it may be.

Now let's review a bit. In most two-person zero sum games a saddle point in pure strategies will not exist because that would say that the players should **always** do the same thing. Such games, which include 2×2 Nim, tic-tac-toe, and many others, are not interesting when played over and over. It seems that if a player should not always play the same strategy, then there should be some randomness involved, because otherwise the opposing player will be able to figure out what the first player is doing and take advantage of it. A player who chooses a pure strategy randomly chooses a row or column according to some probability process that specifies the chance that each pure strategy will be played. These probability vectors are called **mixed strategies**, and will turn out to be the correct class of strategies for each of the players.

Definition 1.3.1 *A mixed strategy is a vector $X = (x_1, \ldots, x_n)$ for player I and $Y = (y_1, \ldots, y_m)$ for player II, where*

$$x_i \geq 0, \sum_{i=1}^{n} x_i = 1 \quad \textit{and} \quad y_j \geq 0, \sum_{j=1}^{m} y_j = 1.$$

The components x_i represent the probability that row i will be used by player I, so $x_i = Prob(I \text{ uses row } i)$, and y_j the probability column j will be used by player II, that is, $y_j = Prob(II \text{ uses row } j)$. Denote the set of mixed strategies with k components by

$$S_k \equiv \{(z_1, z_2, \ldots, z_k) \mid z_i \geq 0, i = 1, 2, \ldots, k, \sum_{i=1}^{k} z_i = 1\}.$$

In this terminology, a mixed strategy for player I is any element $X \in S_n$ and for player II any element $Y \in S_m$. A pure strategy $X \in S_n$ is an element of

Properties of Optimal Strategies (1.3.1)

1. If w is any number such that $E(i, Y) \leq w \leq E(X, j), i = 1, \ldots, n, j = 1, \ldots, m$, where X is a strategy for player I and Y is a strategy for player II, then $w = value(A)$ and (X, Y) must be a saddle point. This is the way to check whether you have a solution to the game. This is part (c) of Theorem 1.3.7 but worth repeating.

2. If X is a strategy for player I and $value(A) \leq E(X, j), j = 1, \ldots, n$, then X is optimal for player I. If Y is a strategy for player II and $value(A) \geq E(i, Y), i = 1, \ldots, m$, then Y is optimal for player II.

3. If Y is optimal for II and $y_j > 0$, then $E(X, j) = value(A)$ for any optimal mixed strategy X for I. Similarly, if X is optimal for I and $x_i > 0$, then $E(i, Y) = value(A)$ for any optimal Y for II. Thus, if any optimal mixed strategy for a player has a strictly positive probability of using a row or a column, then that row or column played against any optimal opponent strategy will yield the value. This result is also called the **Equilibrium Theorem**.

4. If X is any optimal strategy for player I and $E(X, j) > value(A)$ for some column j, then for any optimal strategy Y for player II, we must have $y_j = 0$. Player II would never use column j in any optimal strategy for player II. Similarly, if Y is any optimal strategy for player II and $E(i, Y) < value(A)$, then any optimal strategy X for player I must have $x_i = 0$. If row i for player I gives a payoff when played against an optimal strategy for player II strictly below the value of the game, then player I would never use that row in any optimal strategy for player I.

5. If for any optimal strategy Y for player II, $y_j = 0$, then there is an optimal strategy X for player I so that $E(X, j) > value(A)$. If for any optimal strategy X for I, $x_i = 0$, then there is an optimal strategy Y for II so that $E(i, Y) < value(A)$. This is the converse statement to property 4.

6. If player I has more than one optimal strategy, then player I's set of optimal strategies is a convex, closed, and bounded set. Also, if player II has more than one optimal strategy, then player II's set of optimal strategies is a convex, closed, and bounded set.

$x_i > 0, i = 1, 2, 3$, must be wrong. On the other hand, we know that player I has an optimal strategy of $X^* = (0, 1, 0)$, and so, by the equilibrium theorem (1.3.1), properties 3 and 5, we know that $E(2, Y) = 1$, for an optimal strategy for player II, as well as $E(1, Y) < 1$, and $E(3, Y) < 1$. We need to look for y_1, y_2, y_3 so that

$$y_1 + y_2 + y_3 = 1, \ -2y_1 + 2y_2 - y_3 < 1, 3y_1 + y_3 < 1.$$

We may replace $y_3 = 1 - y_1 - y_2$ and then get a graph of the region of points satisfying all the inequalities in (y_1, y_2) space in Figure 1.5.

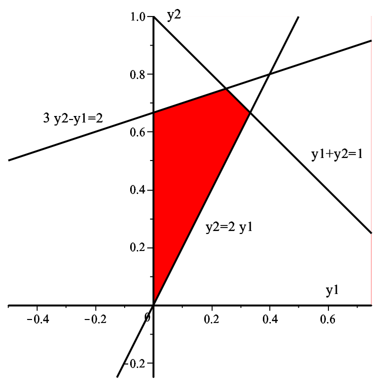


Figure 1.5 Optimal strategy set for Y.

There are lots of points which work. In particular, $Y = (0.15, 0.5, 0.35)$ will give an optimal strategy for player II in which all $y_j > 0$.

1.3.1 Dominated Strategies

Computationally, smaller game matrices are better than large matrices. Sometimes we can reduce the size of the matrix A by eliminating rows or columns (i.e., strategies) that will never be used because there is always a better row or column to use. This is elimination by **dominance.** We should check for dominance whenever we are trying to analyze a game before we begin because it can reduce the size of a matrix.

For example, if every number in row i is bigger than or equal to every corresponding number in row k, specifically $a_{ij} \geq a_{kj}, j = 1, \ldots, m$ (with strict inequality in at least one comparison), then the row player I would never play row k (since she wants the biggest possible payoff), and so we can drop it from the matrix. Similarly, if every number in column j is less than or equal to every corresponding number in column k (i.e., $a_{ij} \leq a_{ik}, i = 1, \ldots, n$), then the column player II would never play column k (since he wants player I to get the smallest possible payoff), and so we can

■ EXAMPLE 1.15

Let's consider

$$A = \begin{bmatrix} -1 & 2 \\ 3 & -4 \\ -5 & 6 \\ 7 & -8 \end{bmatrix}$$

This is a 4×2 game without a saddle point in pure strategies since $v^- = -1, v^+ = 6$. There is also no obvious dominance, so we try to solve the game graphically. Suppose that player II uses the strategy $Y = (y, 1-y)$, then we graph the payoffs $E(i, Y), i = 1, 2, 3, 4,$ as shown in Figure 1.10.

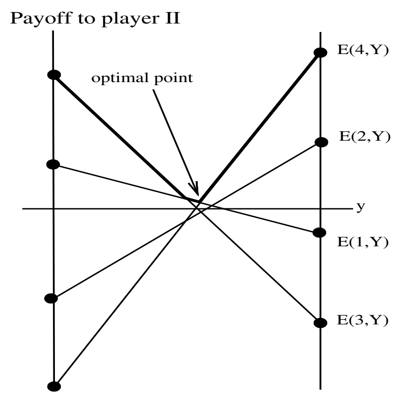


Figure 1.10 Mixed for player II versus 4 rows for player I.

You can see the difficulty with solving games graphically; you have to be very accurate with your graphs. Carefully reading the information, it appears that the optimal strategy for Y will be determined at the intersection point of $E(4, Y) = 7y - 8(1-y)$ and $E(1, Y) = -y + 2(1-y)$. This occurs at the point $y^* = \frac{5}{9}$ and the corresponding value of the game will be $v(A) = \frac{1}{3}$. The optimal strategy for player II is $Y^* = (\frac{5}{9}, \frac{4}{9})$.

Since this uses only rows 1 and 4, we may now drop rows 2 and 3 to find the optimal strategy for player I. In general, we may drop the rows (or columns) not used to get the optimal intersection point. Often that is true because the unused rows are dominated, but not always. To see that here, since $3 \le 7\frac{1}{2} - 1\frac{1}{2}$ and $-4 \le -8\frac{1}{2} + 2\frac{1}{2}$, we see that row 2 is dominated by a convex combination of rows 1 and 4; so row 2 may be dropped. On the other hand, there is no $\lambda \in [0, 1]$ so that $-5 \le 7\lambda - 1(1-\lambda)$ and $6 \le -8\lambda + 2(1-\lambda)$. Row 3 is not dominated by a convex combination of rows 1 and 4, but it is dropped because its payoff line $E(3, Y)$ does not pass through the optimal point.

Considering the matrix using only rows 1 and 4, we now calculate $E(X,1) = -x+7(1-x)$ and $E(X,2) = 2x-8(1-x)$ which intersect at $(x = \frac{5}{6}, v = \frac{1}{3})$. We obtain that row 1 should be used with probability $\frac{5}{6}$ and row 4 should be used with probability $\frac{1}{6}$, so $X^* = (\frac{5}{6}, 0, 0, \frac{1}{6})$. Again, $v(A) = \frac{1}{3}$.

A verification that these are indeed optimal uses Theorem 1.3.7(c). We check that $E(i, Y^*) \leq v(A) \leq E(X^*, j)$ for all rows and columns. This gives

$$\begin{bmatrix} \frac{5}{6} & 0 & 0 & \frac{1}{6} \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 3 & -4 \\ -5 & 6 \\ 7 & -8 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} -1 & 2 \\ 3 & -4 \\ -5 & 6 \\ 7 & -8 \end{bmatrix} \begin{bmatrix} \frac{5}{9} \\ \frac{4}{9} \end{bmatrix} = \begin{bmatrix} \frac{1}{3} \\ -\frac{1}{9} \\ -\frac{1}{9} \\ \frac{1}{3} \end{bmatrix}.$$

Everything checks.

We end this section with a simple analysis of a version of poker, at least a small part of it.

■ EXAMPLE 1.16

This is a modified version of the endgame in poker. Here are the rules. Player I is dealt a card that may be an ace or a king. Player I sees the result but II does not. Player I may then choose to fold or bet. If I folds, he has to pay player II \$1. If I bets, player II may choose to fold or call. If II folds, she pays player I \$1. If player II calls and the card is a king, then player I pays player II \$2, but if the card comes up ace, then player II pays player I \$2.

Why wouldn't player I immediately fold when he gets dealt a king? It is the rule that I must pay II \$1 when I gets a king and he folds. Player I is hoping that player II will fold if I bets while holding a king. This is the element of bluffing, because if II calls while I is holding a king, then I must pay II \$2. Figure 1.11 is a graphical representation of the game.

Now player I has four strategies: FF =fold on ace and fold on king, FB=fold on ace and bet on King, BF =bet on ace and fold on king, and BB =bet on ace and bet on king. Player II has only two strategies, namely, F =fold or C =call.

Assuming that the probability of being dealt a king or an ace is $\frac{1}{2}$ we may calculate the expected reward to player I and get the matrix as follows:

I/II	C	F
FF	-1	-1
FB	$-\frac{3}{2}$	0
BF	$\frac{1}{2}$	0
BB	0	1

.

1.26 Show that for any strategy $X = (x_1, \ldots, x_n) \in S_n$ and any numbers $b_1, \ldots, b_n$, it must be that

$$\max_{X \in S_n} \sum_{i=1}^{n} x_i b_i = \max_{1 \le i \le n} b_i \text{ and } \min_{X \in S_n} \sum_{i=1}^{n} x_i b_i = \min_{1 \le i \le n} b_i.$$

1.27 The properties of optimal strategies (1.3.1) show that $X^* \in S_n$ and $Y^* \in S_m$ are optimal if and only if $\min_j E(X^*, j) = \max_i E(i, Y^*)$. The common value will be the value of the game. Verify this.

1.28 Show that if (X^*, Y^*) and (X^0, Y^0) are both saddle points for the game with matrix A, then so is (X^*, Y^0) and (X^0, Y^*); so is, (X_λ, Y_β) where $X_\lambda = \lambda X^* + (1-\lambda)X^0, Y_\beta = \beta Y^* + (1-\beta)Y^0$ and λ, β any numbers in $[0, 1]$.

1.29 Consider the game with matrix

$$A = \begin{bmatrix} -2 & 3 & 5 & -2 \\ 3 & -4 & 1 & -6 \\ -5 & 3 & 2 & -1 \\ -1 & -3 & 2 & 2 \end{bmatrix}.$$

Someone claims that the strategies $X^* = (\frac{1}{9}, 0, \frac{8}{9}, 0)$ and $Y^* = (0, \frac{7}{9}, \frac{2}{9}, 0)$ are optimal.

(**a**) Is that correct? Why or why not? (**Hint**: Use a previous problem.)

(**b**) If $X^* = (\frac{13}{33}, \frac{5}{33}, 0, \frac{15}{33})$ is optimal and $v(A) = -\frac{26}{33}$, find Y^*.

1.30 In the baseball game Example 1.8 it turns out that an optimal strategy for player I, the batter, is given by $X^* = (x_1, x_2, x_3) = (\frac{2}{7}, 0, \frac{5}{7})$ and the value of the game is $v = \frac{2}{7}$. It is amazing that the batter should never expect a curveball with these payoffs under this optimal strategy. What is the pitcher's optimal strategy Y^*?

1.31 In a football game we use the matrix $A = \begin{bmatrix} 3 & 6 \\ 8 & 0 \end{bmatrix}$. The first row and column represent run, and the second row and column represent pass. The offense is the row player. Column pass means defend against the pass. Use the graphical method to solve this game.

1.6 BEST RESPONSE STRATEGIES

If you are playing a game and you determine, in one way or another, that your opponent is using a particular strategy, or is assumed to use a particular strategy, then what should you do? To be specific, suppose that you are player I and you know, or simply assume, that player II is using the mixed strategy Y, optimal or not for player II. In this case you should play the mixed strategy X that maximizes $E(X, Y)$. This

CHAPTER 2

SOLUTION METHODS FOR MATRIX GAMES

I returned, and saw under the sun, that the race is not to the swift, nor the battle to the strong, ...; but time and chance happeneth to them all.

–Ecclesiates 9:11

2.1 SOLUTION OF SOME SPECIAL GAMES

Graphical methods reveal a lot about exactly how a player reasons her way to a solution, but it is not a very practical method. Now we will consider some special types of games for which we actually have a formula giving the value and the mixed strategy saddle points. Let's start with the easiest possible class of games that can always be solved explicitly and without using a graphical method.

2.1.1 2×2 Games Revisited

We have seen that any 2×2 matrix game can be solved graphically, and many times that is the fastest and best way to do it. But there are also explicit formulas giving the ball, while II anticipates where the ball will be hit. Suppose that II can return a ball hit right 90% of the time, a ball hit left 60% of the time, and a ball hit center 70% of the time. If II anticipates incorrectly, she can return the ball only 20% of the time. Score a return as $+1$ and not return as -1. Find the game matrix and the optimal strategies.

2.3 SYMMETRIC GAMES

Symmetric games are important classes of two-person games in which the players can use the exact same set of strategies and any payoff that player I can obtain using strategy X can be obtained by player II using the same strategy $Y = X$. The two players can switch roles. Such games can be quickly identified by the rule that $A = -A^T$. Any matrix satisfying this is said to be **skew symmetric**. If we want the roles of the players to be symmetric, then we need the matrix to be skew symmetric.

Why is skew symmetry the correct thing? Well, if A is the payoff matrix to player I, then the entries represent the payoffs to player I and the negative of the entries, or $-A$ represent the payoffs to player II. So player II wants to maximize the column entries in $-A$. This means that from player II's perspective, the game matrix must be $(-A)^T$ because it is always the row player by convention who is the maximizer; that is, A is the payoff matrix to player I and $-A^T$ is the payoff to player II. So, if we want the payoffs to player II to be the same as the payoffs to player I, then we must have the same game matrices for each player and so $A = -A^T$. If this is the case, the matrix must be square, $a_{ij} = -a_{ji}$, and the diagonal elements of A, namely, a_{ii}, must be 0. We can say more. In what follows it is helpful to keep in mind that for any appropriate size matrices $(AB)^T = B^T A^T$.

Theorem 2.3.1 *For any skew symmetric game $v(A) = 0$ and if X^* is optimal for player I, then it is also optimal for player II.*

Proof. Let X be any strategy for I. Then

$$E(X,X) = X\ A\ X^T = -X\ A^T\ X^T = -(X\ A^T\ X^T)^T = -X\ A\ X^T = -E(X,X).$$

Therefore $E(X,X) = 0$ and any strategy played against itself has zero payoff.

Let (X^*, Y^*) be a saddle point for the game so that $E(X, Y^*) \le E(X^*, Y^*) \le E(X^*, Y)$, for all strategies (X, Y). Then for any (X, Y), we have

$$E(X,Y) = X\ A\ Y^T = -X\ A^T\ Y^T = -(X\ A^T\ Y^T)^T = -Y\ A\ X^T = -E(Y,X).$$

Hence, from the saddle point definition, we obtain

$$E(X,Y^*) = -E(Y^*,X) \le E(X^*,Y^*) = -E(Y^*,X^*) \le E(X^*,Y) = -E(Y,X^*).$$

Then

$$-E(Y^*,X) \le -E(Y^*,X^*) \le -E(Y,X^*) \Longrightarrow$$
$$E(Y^*,X) \ge E(Y^*,X^*) \ge E(Y,X^*).$$

But this says that Y^* is optimal for player I and X^* is optimal for player II and also that $E(X^*, Y^*) = -E(Y^*, X^*) \Longrightarrow v(A) = 0$. $\square$

EXAMPLE 2.5

General Solution of 3×3 Symmetric Games. For any 3×3 symmetric game we must have

$$A = \begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix}.$$

Any of the following conditions gives a pure saddle point:

1. $a \geq 0, b \geq 0 \Longrightarrow$ saddle at $(1,1)$ position,
2. $a \leq 0, c \geq 0 \Longrightarrow$ saddle at $(2,2)$ position,
3. $b \leq 0, c \leq 0 \Longrightarrow$ saddle at $(3,3)$ position.

Here's why. Let's assume that $a \leq 0, c \geq 0$. In this case if $b \leq 0$ we get $v^- = \max\{\min\{a, b\}, 0, -c\} = 0$ and $v^+ = \min\{\max\{-a, -b\}, 0, c\} = 0$, so there is a saddle in pure strategies at $(2,2)$. All cases are treated similarly. To have a mixed strategy, all three of these must fail.

We next solve the case $a > 0, b < 0, c > 0$ so there is no pure saddle and we look for the mixed strategies.

Let player I's optimal strategy be $X^* = (x_1, x_2, x_3)$. Then

$$\begin{aligned} E(X^*, 1) &= -ax_2 - bx_3 \geq 0 = v(A) \\ E(X^*, 2) &= ax_1 - cx_3 \geq 0 \\ E(X^*, 3) &= bx_1 + cx_2 \geq 0 \end{aligned}$$

Each one is nonnegative since $E(X^*, Y) \geq 0 = v(A)$, for all Y. Now, since $a > 0, b < 0, c > 0$ we get

$$\frac{x_3}{a} \geq \frac{x_2}{-b}, \qquad \frac{x_1}{c} \geq \frac{x_3}{a}, \qquad \frac{x_2}{-b} \geq \frac{x_1}{c}$$

so

$$\frac{x_3}{a} \geq \frac{x_2}{-b} \geq \frac{x_1}{c} \geq \frac{x_3}{a},$$

and we must have equality throughout. Thus, each fraction must be some scalar λ, and so $x_3 = a\,\lambda, x_2 = -b\,\lambda, x_1 = c\,\lambda$. Since they must sum to one, $\lambda = 1/(a - b + c)$. We have found the optimal strategies $X^* = Y^* = (c\,\lambda, -b\,\lambda, a\,\lambda)$. The value of the game, of course is zero.

For example, the matrix

$$A = \begin{bmatrix} 0 & 2 & -3 \\ -2 & 0 & 3 \\ 3 & -3 & 0 \end{bmatrix}$$

2.32 Let $a > 0$. Use the graphical method to solve the game in which player II has an infinite number of strategies with matrix

$$\begin{bmatrix} a & 2a & \frac{1}{2} & 2a & \frac{1}{4} & 2a & \frac{1}{6} & \cdots \\ a & 1 & 2a & \frac{1}{3} & 2a & \frac{1}{5} & 2a & \cdots \end{bmatrix}$$

.

2.33 A Latin square game is a square game in which the matrix A is a Latin square. A Latin square of size n has every integer from 1 to n in each row and column. Solve the game of size 5

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 4 & 1 & 5 & 3 \\ 3 & 5 & 4 & 2 & 1 \\ 4 & 1 & 5 & 3 & 2 \\ 5 & 3 & 2 & 1 & 4 \end{bmatrix}$$

Prove that a Latin square game of size n has $v(A) = (n+1)/2$.

2.34 Two cards, an ace and a jack, are face down on the table. Ace beats jack. Two players start by putting $1 each into a pot. Player I picks a card at random and looks at it. Player I then bets either $2 or $4. Player II can either quit and yield the pot to player I, or **call** the bet (meaning that she may bet the $2 or $4 that player I has chosen the jack). If II calls, she matches I's bet, and then if I holds the ace, player I wins the pot, while if player I has the jack, then player II wins the pot. Find the matrix and solve the game . **Hint**: Player I has 4 strategies, e.g., If ace, bet 2; If jack, bet 2. Player II also has 4 strategies, e.g., If I bets 4, then Fold; If I bets 2, then Call. Player I's strategies are then, $(2,2),(2,4),(4,2),(4,4)$, where the first number is the amount to bet if an ace. Player II's strategies are $(F,C),(C,C),(F,F),(C,F)$, where the first letter is for a bet of 4.

2.35 Two players, Curly and Shemp, are betting on the toss of a fair coin. Shemp tosses the coin, hiding the result from Curly. Shemp looks at the coin. If the coin is heads, Shemp says that he has heads and demands $1 from Curly. If the coin is tails, then Shemp may tell the truth and pay Curly $1, or he may lie and say that he got a head and demands $1 from Curly. Curly can challenge Shemp whenever Shemp demands $1 to see whether Shemp is telling the truth, but it will cost him $2 if it turns out that Shemp was telling the truth. If Curly challenges the call and it turns out that Shemp was lying, then Shemp must pay Curly $2. Find the matrix and solve the game.

2.5 LINEAR PROGRAMMING AND THE SIMPLEX METHOD (OPTIONAL)

Linear programming is one of the major success stories of mathematics for applications. Matrix games of arbitrary size are solved using linear programming, but so is

We need to define a concept of optimal play that should reduce to a saddle point in mixed strategies in the case $B = -A$. It is a fundamental and far-reaching definition due to another genius of mathematics who turned his attention to game theory in the middle twentieth century, John Nash.

Definition 3.1.1 *A pair of mixed strategies $(X^* \in S_n, Y^* \in S_m)$ is a Nash equilibrium if $E_I(X, Y^*) \leq E_I(X^*, Y^*)$ for every mixed $X \in S_n$ and $E_{II}(X^*, Y) \leq E_{II}(X^*, Y^*)$ for every mixed $Y \in S_m$. If (X^*, Y^*) is a Nash equilibrium we denote by $v_A = E_I(X^*, Y^*)$ and $v_B = E_{II}(X^*, Y^*)$ as the optimal payoff to each player. Written out with the matrices, (X^*, Y^*) is a Nash equilibrium if*

$$E_I(X^*, Y^*) = X^* A Y^{*T} \geq X A Y^{*T} = E_I(X, Y^*), \text{ for every } X \in S_n,$$

$$E_{II}(X^*, Y^*) = X^* B Y^{*T} \geq X^* B Y^T = E_{II}(X^*, Y), \text{ for every } Y \in S_m.$$

This says that neither player can gain any expected payoff if either one chooses to deviate from playing the Nash equilibrium, **assuming that the other player is implementing his or her piece of the Nash equilibrium.** On the other hand, if it is known that one player will not be using his piece of the Nash equilibrium, then the other player may be able to increase her payoff by using some strategy other than that in the Nash equilibrium. The player then uses a **best response strategy**. In fact, the definition of a Nash equilibrium says that each strategy in a Nash equilibrium is a best response strategy against the opponent's Nash strategy. Here is a precise definition for two players.

Definition 3.1.2 *A strategy $X^0 \in S_n$ is a* **best response strategy** *to a given strategy $Y^0 \in S_m$ for player II, if*

$$E_I(X^0, Y^0) = \max_{X \in S_n} E_I(X, Y^0).$$

Similarly, a strategy $Y^0 \in S_m$ is a **best response strategy** *to a given strategy $X^0 \in S_n$ for player I, if*

$$E_{II}(X^0, Y^0) = \max_{Y \in S_m} E_{II}(X^0, Y).$$

In particular, another way to define a Nash equilibrium (X^*, Y^*) is that X^* maximizes $E_{\text{I}}(X, Y^*)$ over all $X \in S_n$ and Y^* maximizes $E_{\text{II}}(X^*, Y)$ over all $Y \in S_m$. X^* is a best response to Y^* and Y^* is a best response to X^*.

If $B = -A$, a bimatrix game is a zero sum two-person game and a Nash equilibrium is the same as a saddle point in mixed strategies. It is easy to check that from the definitions because $E_{\text{I}}(X, Y) = XAY^T = -E_{\text{II}}(X, Y)$.

Note that a Nash equilibrium in pure strategies will be a row i^* and column j^* satisfying

$$a_{ij^*} \leq a_{i^*j^*} \text{ and } b_{i^*j} \leq b_{i^*j^*}, i = 1, \ldots, n, j = 1, \ldots, m.$$

Then $v(A) = \frac{3}{2}$ is the safety value for player I and $v(B^T) = \frac{3}{4}$ is the safety value for player II.

The maxmin strategy for player I is $X = (\frac{3}{4}, \frac{1}{4})$, and the implementation of this strategy guarantees that player I can get at least her safety level. In other words, if I uses $X = (\frac{3}{4}, \frac{1}{4})$, then $E_{\text{I}}(X, Y) \geq v(A) = \frac{3}{2}$ no matter what Y strategy is used by II. In fact

$$E_{\text{I}}\left(\left(\frac{3}{4}, \frac{1}{4}\right), Y\right) = \frac{3}{2}(y_1 + y_2) = \frac{3}{2}, \text{ for any strategy } Y = (y_1, y_2).$$

The maxmin strategy for player II is $Y = X^{B^T} = (\frac{3}{4}, \frac{1}{4})$, which she can use to get at least her safety value of $\frac{3}{4}$.

Is their a connection between the safety levels and a Nash equilibrium? The safety levels are the guaranteed amounts each player can get by using their own individual maxmin strategies, so any rational player must get at least the safety level in a bimatrix game. In other words, it has to be true that if (X^*, Y^*) is a Nash equilibrium for the bimatrix game (A, B), then

$$E_{\text{I}}(X^*, Y^*) = X^* A Y^{*T} \geq value(A) \text{ and } E_{\text{II}}(X^*, Y^*) = X^* B Y^{*T} \geq value(B^T).$$

This would say that in the bimatrix game, if players use their Nash points, they get at least their safety levels. That's what it means to be **individually rational**.

Here's why that's true.

Proof. It's really just from the definitions. The definition of Nash equilibrium says

$$E_{\text{I}}(X^*, Y^*) = X^* A Y^{*T} \geq E_{\text{I}}(X, Y^*) = X A Y^{*T}, \text{ for all } X \in S_n.$$

But if that is true for all mixed X, then

$$E_{\text{I}}(X^*, Y^*) \geq \max_{X \in S_n} X A Y^{*T} \geq \min_{Y \in S_m} \max_{X \in S_n} X A Y^T = value(A).$$

The other part of a Nash definition gives us

$$\begin{aligned} E_{\text{II}}(X^*, Y^*) = X^* B Y^{*T} &\geq \max_{Y \in S_m} X^* B Y^T \\ &= \max_{Y \in S_m} Y B^T X^{*T} \qquad (\text{since } X^* B Y^T = Y B^T X^{*T}) \\ &\geq \min_{X \in S_n} \max_{Y \in S_m} Y B^T X^T = value(B^T). \end{aligned}$$

Each player does at least as well as assuming the worst. □

PROBLEMS

3.1 Show that (X^*, Y^*) is a saddle point of the game with matrix A if and only if (X^*, Y^*) is a Nash equilibrium of the bimatrix game $(A, -A)$.

2. $R = 0, r > 0$. Solutions are $0 \le x \le 1, y = 0$.

3. $R = 0, r < 0$. Solutions are $0 \le x \le 1, y = 1$.

4. $R > 0$. Solutions are

$$\begin{aligned} &\text{if } y = 0 \implies 0 \le x \le \frac{r}{R}, \\ &\text{if } 0 < y < 1 \implies x = \frac{r}{R}, \\ &\text{if } y = 1 \implies 1 \ge x \ge \frac{r}{R}. \end{aligned}$$

5. $R < 0$. In this final case the set of all possible solutions are

$$\begin{aligned} &\text{if } y = 0 \implies 1 \ge x \ge \frac{r}{R}, \\ &\text{if } 0 < y < 1 \implies x = \frac{r}{R}, \\ &\text{if } y = 1 \implies 0 \le x \le \frac{r}{R}. \end{aligned}$$

We may also draw a zigzag line for player II that will be a graph of the rational reaction set for player II to a given strategy X for player I. For example, if $R > 0$, the graph would look like Figure 3.2. This bold line would be a graph of the rational

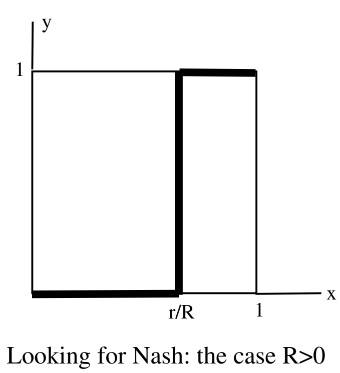


Figure 3.2 Rational reaction set for player II.

reaction set for player II against a given X. It has the explicit representation in the case $R > 0$ given by

$$R_{\text{II}} = \left\{(x, 0) \mid 0 \le x \le \frac{r}{R}\right\} \cup \left\{\left(\frac{r}{R}, y\right) \mid 0 < y < 1\right\} \cup \left\{(x, 1) \mid \frac{r}{R} \le x \le 1\right\}.$$

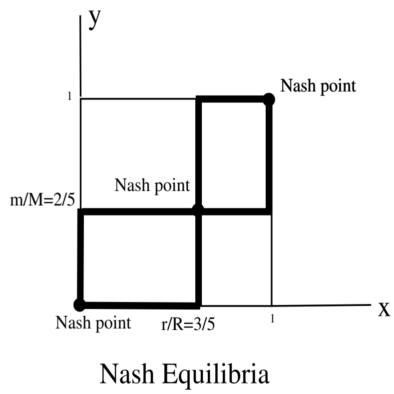


Figure 3.3 Rational reaction sets for both players

This is curious because the expected payoffs to each player are **much less** than they could get at the other Nash points.

We will see pictures like Figure 3.3 again in the next section when we consider an easier way to get Nash equilibria.

Remark: A direct way to calculate the rational reaction sets for 2×2 **games.** This is a straightforward derivation of the rational reaction sets for the bimatrix game with matrices (A, B). Let $X = (x, 1-x),\ Y = (y, 1-y)$ be any strategies and define

$$f(x,y) = E_{\mathrm{I}}(X,Y) \ \text{ and } \ g(x,y) = E_{\mathrm{II}}(X,Y).$$

The idea is to find for a fixed $0 \le y \le 1$, the best response to y. Accordingly,

$$\begin{aligned}
\max_{0\le x\le 1} f(x,y) &= \max_{0\le x\le 1} xE_{\mathrm{I}}(1,Y) + (1-x)E_{\mathrm{I}}(2,Y) \\
&= x[E_{\mathrm{I}}(1,Y) - E_{\mathrm{I}}(2,Y)] + E_{\mathrm{I}}(2,Y) \\
&= \begin{cases} E_{\mathrm{I}}(2,Y) & \text{at } x = 0 \text{ if } E_{\mathrm{I}}(1,Y) < E_{\mathrm{I}}(2,Y); \\ E_{\mathrm{I}}(1,Y) & \text{at } x = 1 \text{ if } E_{\mathrm{I}}(1,Y) > E_{\mathrm{I}}(2,Y); \\ E_{\mathrm{I}}(2,Y) & \text{at any } 0 < x < 1 \text{ if } E_{\mathrm{I}}(1,Y) = E_{\mathrm{I}}(2,Y). \end{cases}
\end{aligned}$$

Now we have to consider the inequalities in the conditions. For example,

$$E_{\mathrm{I}}(1,Y) < E_{\mathrm{I}}(2,Y) \Leftrightarrow My < m,\ M = a_{11} - a_{12} - a_{21} + a_{22},\ m = a_{22} - a_{12}.$$

If $M > 0$ this is equivalent to the condition $0 \le y < m/M$. Consequently, in the case $M > 0$, the best response to any $0 \le y < M/m$ is $x = 0$. All remaining cases components will be greater than one. You have to be careful in trying to do this with Maple because solving the system of equations doesn't always work, especially if there is more than one interior Nash equilibrium (which could occur for matrices that have more than two pure strategies).

EXAMPLE 3.11

Here is a last example in which the equations do not work (see problem 3.11) because it turns out that one of the columns should never be played by player II. That means that the mixed Nash is not in the interior, but on the boundary of $S_n \times S_m$.

Let's consider the game with payoff matrices

$$A = \begin{bmatrix} 2 & 3 & 4 \\ 0 & 4 & 3 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 & 1 & -1 \\ 2 & 0 & 1 \end{bmatrix}.$$

You can calculate that the safety levels are $value(A) = 2$, with pure saddle $X_A = (1,0), Y_A = (1,0,0)$, and $value(B^T) = \frac{2}{3}$, with saddle $X_B = (\frac{1}{3}, \frac{2}{3}, 0), Y_B = (\frac{2}{3}, \frac{1}{3})$. These are the amounts that each player can get assuming that both are playing in a zero sum game with the two matrices.

Now, let $X = (x, 1-x), Y = (y_1, y_2, 1 - y_1 - y_2)$ be a Nash point for the bimatrix game. Calculate

$$E_1(x, y_1, y_2) = XAY^T = x[y_1 - 2y_2 + 1] + y_2 - 3y_1 + 3.$$

Player I wants to maximize $E_1(x, y_1, y_2)$ for given fixed $y_1, y_2 \in [0,1], y_1 + y_2 \leq 1$, using $x \in [0,1]$. So we look for $\max_x E_1(x, y_1, y_2)$. For fixed y_1, y_2, we see that $E_1(x, y_1, y_2)$ is a straight line with slope $y_1 - 2y_2 + 1$. The maximum of that line will occur at an endpoint depending on the sign of the slope. Here is what we get:

$$\begin{aligned}\max_{0 \leq x \leq 1} x[y_1 - 2y_2 + 1] \quad & + y_2 - 3y_1 + 3 \\ &= \begin{cases} -2y_1 - y_2 + 4 & \text{if } y_1 > 2y_2 - 1; \\ y_2 - 3y_1 + 3 & \text{if } y_1 = 2y_2 - 1; \\ y_2 - 3y_1 + 3 & \text{if } y_1 < 2y_2 - 1. \end{cases} \\ &= \begin{cases} E_1(1, y_1, y_2) & \text{if } y_1 > 2y_2 - 1; \\ E_1(x, 2y_2 - 1, y_2) & \text{if } y_1 = 2y_2 - 1; \\ E_1(0, y_1, y_2) & \text{if } y_1 < 2y_2 - 1. \end{cases}\end{aligned}$$

Along any point of the straight line $y_1 = 2y_2 - 1$ the maximum of $E_1(x, y_1, y_2)$ is achieved at any point $0 \leq x \leq 1$. We end up with the following set of because $XJ_n^T = J_m Y^T = 1$. But this is exactly what it means to be a Nash point. This means that (X^*, Y^*) is a Nash point if and only if

$$X^* A Y^{*T} J_n^T \geq A Y^{*T}, \quad (X^* B Y^{*T}) J_m \geq X^* B.$$

We have already seen this in Proposition 3.2.3.

Now suppose that (X^*, Y^*) is a Nash point. We will see that if we choose the scalars

$$p^* = E_{\mathrm{I}}(X^*, Y^*) = X^* A Y^{*T} \text{ and } q^* = E_{\mathrm{II}}(X^*, Y^*) = X^* B Y^{*T},$$

then (X^*, Y^*, p^*, q^*) is a solution of the nonlinear program. To see this, we first show that all the constraints are satisfied. In fact, by the equivalent characterization of a Nash point we just derived, we get

$$X^* A Y^{*T} J_n^T = p^* J_n^T \geq A Y^{*T} \quad \text{and} \quad (X^* B Y^{*T}) J_m = q^* J_m \geq X^* B.$$

The rest of the constraints are satisfied because $X^* \in S_n$ and $Y^* \in S_m$. In the language of nonlinear programming, we have shown that (X^*, Y^*, p^*, q^*) is a **feasible point**. The **feasible set** is the set of all points that satisfy the constraints in the nonlinear programming problem.

We have left to show that (X^*, Y^*, p^*, q^*) maximizes the objective function

$$f(X, Y, p, q) = XAY^T + XBY^T - p - q$$

over the set of the possible feasible points.

Since every feasible solution (meaning it maximizes the objective over the feasible set) to the nonlinear programming problem must satisfy the constraints $AY^T \leq pJ_n^T$ and $XB \leq qJ_m$, multiply the first on the left by X and the second on the right by Y^T to get

$$XAY^T \leq pXJ_n^T = p, \qquad XBY^T \leq qJ_m Y^T = q.$$

Hence, any **possible** solution gives the objective

$$f(X, Y, p, q) = XAY^T + XBY^T - p - q \leq 0.$$

So $f(X, Y, p, q) \leq 0$ for any feasible point. But with $p^* = X^* A Y^{*T}, q^* = X^* B Y^{*T}$, we have seen that (X^*, Y^*, p^*, q^*) is a feasible solution of the nonlinear programming problem and

$$f(X^*, Y^*, p^*, q^*) = X^* A Y^{*T} + X^* B Y^{*T} - p^* - q^* = 0$$

by definition of p^* and q^*. Hence this point (X^*, Y^*, p^*, q^*) both is feasible and gives the maximum objective (which we know is zero) over any possible feasible solution and so is a solution of the nonlinear programming problem. This shows that if we have a Nash point, it must solve the nonlinear programming problem.

3.22 Since every two-person **zero sum** game can be formulated as a bimatrix game, show how to modify the Lemke–Howson algorithm to be able to calculate saddle points of zero sum two–person games. Then use that to find the value and saddle point for the game with matrix

$$A = \begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & -3 \\ 4 & -3 & 5 \\ -3 & \frac{1}{2} & -9 \end{bmatrix}.$$

Check your answer by using the linear programming method to solve this game.

3.23 Consider the following bimatrix for a version of the game of chicken (see Problem 3.3):

I/II	Straight	Avoid
Straight	$(1,1)$	$(-1,2)$
Avoid	$(2,-1)$	$(-3,-3)$

(**a**) What is the explicit objective function for use in the Lemke–Howson algorithm?
(**b**) What are the explicit constraints?
(**c**) Solve the game.

3.24 Use nonlinear programming to find all Nash equilibria for the game and expected payoffs for the game with bimatrix

$$\begin{bmatrix} (1,2) & (0,0) & (2,0) \\ (0,0) & (2,3) & (0,0) \\ (2,0) & (0,0) & (-1,6) \end{bmatrix}$$

3.25 Consider the game with bimatrix

$$\begin{bmatrix} (-3,-4) & (2,-1) & (0,6) & (1,1) \\ (2,0) & (2,2) & (-3,0) & (1,-2) \\ (2,-3) & (-5,1) & (-1,-1) & (1,-3) \\ (-4,3) & (2,-5) & (1,2) & (-3,1) \end{bmatrix}$$

There are six Nash equilibria for this game. Find as many as you can by adjusting `initialpoint=` in the Maple commands.

3.26 Consider the gun duel between Pierre and Bill. Modify the payoff functions so that it becomes a noisy duel with $a_1 = -2, b_1 = -1, c_1 = 2, d_1 = -1, e_1 = 1, f_1 = 2, g_1 = -2, h_1 = 1, k_1 = -1, \ell_1 = 2$, for player I, and $a_2 = -1, b_2 = 1, c_2 = 1, d_2 = 1, e_2 = -1, f_2 = 1, g_2 = 0, h_2 = -1, k_2 = 1, \ell_2 = 1$, for player II. Then solve the game and obtain at least one mixed Nash equilibrium.

3. From the previous two steps you end up with three equations involving q_1, q_2^+, q_2^-. Treat these as three equations in three unknowns and solve.

For example, let's take the price function $P(q) = \Gamma - q, 0 \le q \le \Gamma$.

1. Firm 2 has the cost of production $c^+ q_2$ with probability p and the cost of production $c^- q_2$ with probability $1 - p$. Firm 2 will solve the problem for each cost c^+, c^- assuming that q_1 is known:

$$\max_{q_2} q_2(\Gamma - (q_1 + q_2) - c^+) \implies q_2^+ = \frac{1}{2}(\Gamma - q_1 - c^+)$$

$$\max_{q_2} q_2(\Gamma - (q_1 + q_2) - c^-) \implies q_2^- = \frac{1}{2}(\Gamma - q_1 - c^-)$$

2. Next, firm 1 will maximize the expected profit using the two quantities q_2^+, q_2^-. Firm 1 seeks the production quantity q_1, which solves

$$\max_{q_1} q_1[\Gamma - (q_1 + q_2^+) - c_1]p + q_1[\Gamma - (q_1 + q_2^-) - c_1](1 - p).$$

This is maximized at

$$q_1 = \frac{1}{2}[p(\Gamma - q_2^+ - c_1) + (1 - p)(\Gamma - q_2^- - c_1)].$$

3. Summarizing, we now have the following system of equations for the variables q_1, q_2^-, q_2^+:

$$\begin{aligned} q_2^+ &= \frac{1}{2}(\Gamma - q_1 - c^+), \\ q_2^- &= \frac{1}{2}(\Gamma - q_1 - c^-), \\ q_1 &= \frac{1}{2}[p(\Gamma - q_2^+ - c_1) + (1 - p)(\Gamma - q_2^- - c_1)]. \end{aligned}$$

Solving these, we finally arrive at the optimal production levels:

$$\begin{aligned} q_1^* &= \frac{1}{3}[\Gamma - 2c_1 + pc^+ + (1 - p)c^-], \\ q_2^{+*} &= \frac{1}{3}[\Gamma + c_1] - \frac{1}{6}[(1 - p)c^- + pc^+] - \frac{1}{2}c^+, \\ q_2^{-*} &= \frac{1}{3}[\Gamma - 2c^- + c_1] + \frac{1}{6}p(c^- - c^+). \end{aligned}$$

Notice that if we require that the production levels be nonnegative, we need to put some conditions on the costs and Γ.

firm, say, firm 1, who will announce its production quantity publicly. Then firm 2 will decide how much to produce. In other words, given that one firm knows the production quantity of the other, determine how much each will or should produce.

Suppose that firm 1 announces that it will produce q_1 gadgets at cost c_1 dollars per unit. It is then up to firm 2 to decide how many gadgets, say, q_2 at cost c_2, it will produce. We again assume that the unit costs are constant. The price per unit will then be considered a function of the total quantity produced so that $p = p(q_1, q_2) = (\Gamma - q_1 - q_2)^+ = \max\{\Gamma - q_1 - q_2, 0\}$. The profit functions will be

$$u_1(q_1, q_2) = (\Gamma - q_1 - q_2)q_1 - c_1 q_1,$$
$$u_2(q_1, q_2) = (\Gamma - q_1 - q_2)q_2 - c_2 q_2.$$

These are the same as in the simplest Cournot model, but now q_1 is fixed as given. It is not variable when firm 1 announces it. So what we are really looking for is the best response of firm 2 to the production announcement q_1 by firm 1. In other words, firm 2 wants to know how to choose $q_2 = q_2(q_1)$ so as to

Maximize over q_2, given q_1, the function $u_2(q_1, q_2(q_1))$.

This is given by calculus as

$$q_2(q_1) = \frac{\Gamma - q_1 - c_2}{2}.$$

This is the amount that firm 2 should produce when firm 1 announces the quantity of production q_1.

Now, firm 1 has some clever employees who know calculus and game theory and can perform this calculation as well as we can. Firm 1 knows what firm 2's optimal production quantity should be, given its own announcement of q_1. Therefore, firm 1 should choose q_1 to maximize its own profit function knowing that firm 2 will use production quantity $q_2(q_1)$:

$$\begin{aligned} u_1(q_1, q_2(q_1)) &= q_1(\Gamma - q_1 - q_2(q_1)) - c_1 q_1 \\ &= q_1\left(\Gamma - q_1 - \frac{\Gamma - q_1 - c_2}{2}\right) - c_1 q_1 \\ &= q_1 \frac{\Gamma - q_1}{2} + q_1\left(\frac{c_2}{2} - c_1\right). \end{aligned}$$

Firm 1 wants to choose q_1 to make this as large as possible. By calculus, we find that

$$q_1^* = \frac{\Gamma - 2c_1 + c_2}{2}, \quad \text{and} \quad u_1(q_1^*, q_2^*) = \frac{(\Gamma - 2c_1 + c_2)^2}{8}.$$

Then the optimal production quantity for firm 2 will be

$$q_2^* = q_2(q_1^*) = \frac{\Gamma + 2c_1 - 3c_2}{4}.$$

The equilibrium profit function for firm 2 is then

$$u_2(q_1^*, q_2^*) = \frac{(\Gamma + 2c_1 - 3c_2)^2}{16},$$

and for firm 1, it is

$$u_1(q_1^*, q_2^*) = \frac{(\Gamma - 2c_1 + c_2)^2}{8}.$$

For comparison, we will set $c_1 = c_2 = c$ and then recall the optimal production quantities for the Cournot model:

$$q_1^c = \frac{\Gamma - 2c_1 + c_2}{3} = \frac{\Gamma - c}{3}, \qquad q_2^c = \frac{\Gamma + c_1 - 2c_2}{3} = \frac{\Gamma - c}{3}.$$

The equilibrium profit functions were

$$u_1(q_1^c, q_2^c) = \frac{(\Gamma + c_2 - 2c_1)^2}{9} = \frac{(\Gamma - c)^2}{9},$$
$$u_2(q_1^c, q_2^c) = \frac{(\Gamma + c_1 - 2c_2)^2}{9} = \frac{(\Gamma - c)^2}{9}.$$

In the Stackelberg model we have

$$q_1^* = \frac{\Gamma - c}{2} > q_1^c, \quad q_2^* = \frac{\Gamma - c}{4} < q_2^c.$$

So firm 1 produces more and firm 2 produces less in the Stackelberg model than if firm 2 did not have the information announced by firm 1. For the firm's profits, we have

$$u_1(q_1^c, q_2^c) = \frac{(\Gamma - c)^2}{9} < u_1(q_1^*, q_2^*) = \frac{(\Gamma - c)^2}{8},$$

$$u_2(q_1^c, q_2^c) = \frac{(\Gamma - c)^2}{9} > u_2(q_1^*, q_2^*) = \frac{(\Gamma - c)^2}{16}.$$

Firm 1 makes more money by announcing the production level, and firm 2 makes less with the information.

One last comparison is the total quantity produced

$$q_1^c + q_2^c = \frac{2}{3}(\Gamma - c) < q_1^* + q_2^* = \frac{3\Gamma - 2c_1 - c_2}{4} = \frac{3}{4}(\Gamma - c)$$

and the price at equilibrium (recall that $\Gamma > c$):

$$P(q_1^* + q_2^*) = \frac{\Gamma + 3c}{4} < P(q_1^c + q_2^c) = \frac{\Gamma + 2c}{3}.$$

As long as $u_2(q_2^*) \geq 0$, firm 2 has an incentive to enter the market. If we interpret the constant b as a fixed cost to enter the market, this will require that

$$\frac{(\Gamma - a)^2}{16} > b,$$

or else firm 2 cannot make a profit.

Now here is a more serious analysis because firm 1 is not about to sit by idly and let another firm enter the market. Therefore, firm 1 will now analyze the Cournot model assuming that there is a firm 2 against which firm 1 is competing. Firm 1 looks at the profit function for firm 2:

$$u_2(q_1, q_2) = (\Gamma - q_1 - q_2)q_2 - (aq_2 + b),$$

and maximizes this as a function of q_2 to get

$$q_2^m = \frac{\Gamma - q_1 - a}{2} \text{ and } u_2(q_1, q_2^m) = \frac{(\Gamma - q_1 - a)^2}{4} - b$$

as the maximum profit to firm 2 if firm 1 produces q_1 gadgets. Firm 1 reasons that it can set q_1 so that firm 2's profit is zero:

$$u_2(q_1, q_2^m) = \frac{(\Gamma - q_1 - a)^2}{4} - b = 0 \implies q_1^0 = \Gamma - 2\sqrt{b} - a.$$

This gives a zero profit to firm 2. Consequently, if firm 1 decides to produce q_1^0 gadgets, firm 2 has no incentive to enter the market. The price at this quantity will be

$$D(q_1^0) = \Gamma - (\Gamma - 2\sqrt{b} - a) = 2\sqrt{b} + a,$$

and the profit for firm 1 at this level of production will be

$$u_1(q_1^0) = (\Gamma - q_1^0)q_1^0 - (aq_1^0 + b) = 2\sqrt{b}(\Gamma - a) - 5b.$$

This puts a requirement on Γ that $\Gamma > a + \frac{5}{2}\sqrt{b}$, or else firm 1 will also make a zero profit.

PROBLEMS

4.1 Suppose instead of two firms in the Cournot model with payoff functions (4.2.1), that there are N firms. Formulate this model and find the optimal quantities each of the N firms should produce. Instead of a duopoly, this is an oligopoly. What happens when the firms all have the same costs and $N \to \infty$?

4.2 Two firms produce identical products. The market price for total production quantity q is $P(q) = 100 - 2\sqrt{q}$. Firm 1's production cost is $C_1(q_1) = q_1 + 10$, and firm 2's production cost is $C_2(q_2) = 2q_2 + 5$. Find the profit functions and the Nash equilibrium quantities of production and profits.

4.3 Compare profits for firm 1 in the model with uncertain costs and the standard Cournot model. Assume $\Gamma = 15, c_1 = 4, c^+ = 5, c^- = 1$ and $p = 0.5$.

4.4 Suppose that we consider the Cournot model with uncertain costs but with three possible costs, $Prob(C_2 = c^i) = r_i,\ i = 1, 2, 3$, where $r_i \geq 0, r_1 + r_2 + r_3 = 1$. Solve for the optimal production quantities. Find the explicit production quantities when $r_1 = \frac{1}{2}, r_2 = \frac{1}{8}, r_3 = \frac{3}{8}, \Gamma = 100$, and $c_1 = 2, c^1 = 1, c^2 = 2, c^3 = 5$.

4.5 In the Stackelberg model compare the quantity produced, the profit, and the prices for firm 1 assuming that firm 2 did not exist so that firm 1 is a monopolist.

4.6 Suppose that two firms have constant unit costs $c_1 = 2, c_2 = 1$ and $\Gamma = 19$ in the Stackelberg model.
(**a**) What are the profit functions?
(**b**) How much should firm 2 produce as a function of q_1?
(**c**) How much should firm 1 produce? (**d**) How much, then, should firm 2 produce?

4.7 Set up and solve a Stackelberg model given three firms with constant unit costs c_1, c_2, c_3 and firm 1 announcing production quantity q_1.

4.8 In the Bertrand model show that if $c_1 = c_2 = c$, then $(p_1^*, p_2^*) = (c, c)$ is a Nash equilibrium.

4.9 Determine the entry deterrence level of production for firm 1 given $\Gamma = 100, a = 2, b = 10$. How much profit is lost by setting the price to deter a competitor?

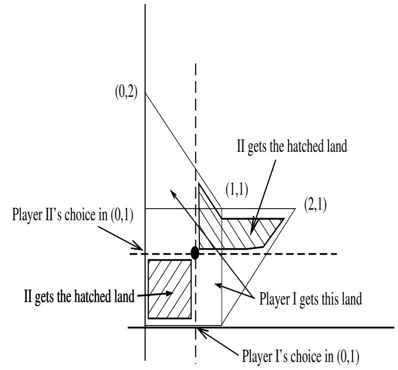
4.10 We could make one more adjustment in the Bertrand model and see what effect it has on the model. What if we put a limit on the total quantity that a firm can produce? This limits the supply and possibly will put a floor on prices. Let $K \geq \frac{\Gamma}{2}$ denote the maximum quantity of gadgets that each firm can produce and recall that $D(p) = \Gamma - p$ is the quantity of gadgets demanded at price p. Find the profit functions for each firm.

4.11 Suppose that the demand functions in the Bertrand model are given by

$$q_1 = D_1(p_1, p_2) = (a - p_1 + bp_2)^+ \text{ and } q_2 = D_2(p_1, p_2) = (a - p_2 + bp_1)^+,$$

where $1 \geq b > 0$. This says that the quantity of gadgets sold by a firm will increase if the price set by the opposing firm is too high. Assume that both firms have a cost of production $c \leq \min\{p_1, p_2\}$.
(**a**) Show that the profit functions will be given by

$$u_i(p_1, p_2) = D_i(p_1, p_2)(p_i - c), \quad i = 1, 2.$$



Player I chooses a vertical line between $(0, 1)$ on the x axis and player II chooses a horizontal line between $(0, 1)$ on the y axis. Player I gets the land below II's choice and right of I's choice as well as the land above II's choice and left of I's line. Player II gets the rest of the land. Both players want to choose their line so as to maximize the amount of land they get. Formulate this as a game with continuous strategies and solve it.

4.16 Two countries are at war over a piece of land. Country 1 places value v_1 on the land, and country 2 values it at v_2. The players choose the time at which to concede the land to the other player, but there is a cost for letting time pass. Suppose that the cost to each country is $C_i, i = 1, 2$ per unit of time. The first player to concede yields the land to the other player at that time. If they concede at the same time, each player gets half the land. Determine the payoffs to each player and determine the pure Nash equilibria. Consider the case $C_1 = C_2$.

4.17 Two countries are at war over an asset with a total value of $V > 0$. Country 1 allocates an effort $x > 0$, and country 2 allocates an effort $y > 0$ to acquire all or a portion of V. The portion of V won by country 1 if they allocate effort x is $(x/(x + y))V$ at cost C_1x, where $C_1 > 0$ is a constant. Similarly, the portion of V won by country 2 if they allocate effort y is $(y/(x+y))V$ at cost C_2y, where $C_2 > 0$ is a constant. The total reward to each country is then

$$u_1(x, y) = V\frac{x}{x + y} - C_1x \quad \text{and} \quad u_2(x, y) = V\frac{y}{x + y} - C_2y, \;\; x > 0, y > 0.$$

Show that these payoff functions are concave in the variable they control and then find the Nash equilibrium using calculus.

4.18 Corn is a food product in high demand but also enjoys a government price subsidy. Assume that the demand for corn (in bushels) is given by $D(p) = 150000(15 - p)^+$, where p is the price per bushel. The government program guarantees that $p \geq 2$. Suppose that there are three corn producers who have reaped 1 million bushels each. They each have the choice of how much to send to market and

Why? Well, $v(123) = d$ because the car will be sold for d, $v(1) = M$ because the car is worth M to player 1, $v(13) = d$ because player 1 will sell the car to player 3 for $d > M$, $v(12) = c$ because the car will be sold to player 2 for $c > M$, and so on. The reader can easily check that v is a characteristic function.

3. A customer wants to buy a bolt and a nut for the bolt. There are three players but player 1 owns the bolt and players 2 and 3 each own a nut. A bolt together with a nut is worth 5. We could define a characteristic function for this game as

$$v(123) = 5,\ v(12) = v(13) = 5,\ \ v(1) = v(2) = v(3) = 0, \quad \text{and} \quad v(\emptyset) = 0.$$

In contrast to the car problem $v(1) = 0$ because a bolt without a nut is worthless to player 1.

4. A small research drug company, labeled 1, has developed a drug. It does not have the resources to get FDA (Food and Drug Administration) approval or to market the drug, so it considers selling the rights to the drug to a big drug company. Drug companies 2 and 3 are interested in buying the rights but only if both companies are involved in order to spread the risks. Suppose that the research drug company wants $1 billion, but will take $100 million if only one of the two big drug companies are involved. The profit to a participating drug company 2 or 3 is $5 billion, which they split. Here is a possible characteristic function with units in billions:

$$v(1) = v(2) = v(3) = 0, v(12) = 0.1, v(13) = 0.1, v(23) = 0, v(123) = 5,$$

because any coalition which doesn't include player 1 will be worth nothing.

5. A **simple game** is one in which $v(S) = 1$ or $v(S) = 0$ for all coalitions S. A coalition with $v(S) = 1$ is called a **winning coalition** and one with $v(S) = 0$ is a **losing coalition**. For example, if we take $v(S) = 1$ if $|S| > n/2$ and $v(S) = 0$ otherwise, we have a simple game that is a model of majority voting. If a coalition contains more than half of the players, it has the majority of votes and is a winning coalition.

6. In any bimatrix (A, B) nonzero sum game we may obtain a characteristic function by taking $v(1) = value(A)$, $v(2) = value(B^T)$, and $v(12) =$sum of largest payoff pair in (A, B). Checking that this is a characteristic function is skipped. The next example works one out.

■ **EXAMPLE 5.2**

In this example we will construct a characteristic function for a version of the prisoner's dilemma game in which we assumed that there was no cooperation. Now we will assume that the players may cooperate and negotiate. One form and conversely. For example, one such possible game is $S = \{12\}$ versus $N - S = 3$, in which player $S = \{12\}$ is the row player and player 3 is the column player We also have to consider the game 3 versus $\{12\}$, in which player 3 is the row player and coalition $\{12\}$ is the column player. So now we go through the construction.

1. **Play** $S = \{12\}$ **versus** $\{3\}$**.** players 1 and 2 team up against player 3. We first write down the associated matrix game.

12 versus 3		player 3	
		A	B
player 12	AA	2	−2
	AB	2	1
	BA	3	2
	BB	4	3

For example, if 1 plays A and 2 plays A and 3 plays B, the payoffs in the nonzero sum game are $(-3, 1, 2)$ and so the payoff to player 12 is $-3 + 1 = -2$, the **sum of the payoff to player 1 and player 2**, which is our coalition. Now we calculate the value of the zero sum two-person game with this matrix to get the $value(12 \text{ vs. } 3) = 3$ and we write $v(12) = 3$. This is the maximum possible guaranteed benefit to coalition $\{12\}$ because it even assumes that player 3 is actively working against the coalition.

In the game $\{3\}$ versus $\{12\}$ we have 3 as the row player and players $\{12\}$ as the column player. We now want to know the maximum possible payoff to player 3 assuming that the coalition $\{12\}$ is actively working against player 3. The matrix is

3 versus 12		12			
		AA	AB	BA	BB
3	A	0	2	−1	−1
	B	2	1	−1	−1

The value of this game is -1. Consequently, in the game $\{3\}$ versus $\{12\}$ we would get $v(3) = -1$. Observe that the game matrix for 3 versus 12 is **not** the transpose of the game matrix for 12 versus 3.

2. **Play** $S = \{13\}$ **versus** $\{2\}$. The game matrix is

13 versus 2		player 2	
		A	B
player 13	AA	1	6
	AB	−1	1
	BA	0	2
	BB	1	1

We see that the value of this game is 1 so that $v(13) = 1$. In the game $\{2\}$ versus $\{13\}$ we have $\boxed{2}$ as the row player and the matrix

$\boxed{2}$ versus $\boxed{13}$		$\boxed{13}$			
		AA	AB	BA	BB
$\boxed{2}$	A	1	1	2	0
	B	-2	1	1	1

The value of this game is $\frac{1}{4}$, and so $v(2) = \frac{1}{4}$.

Continuing in this way, we summarize that the characteristic function for this three–person game is

$$v(1) = 1,\ v(2) = \frac{1}{4},\ v(3) = -1,$$
$$v(12) = 3,\ v(13) = 1,\ v(23) = 1,$$
$$v(123) = 4,\ v(\emptyset) = 0.$$

The value $v(123) = 4$ is obtained by taking the largest sum of the payoffs that they would achieve if they all cooperated. This number is obtained from the pure strategies: 3 plays A, 1 plays A, and 2 plays B with payoffs $(4, -2, 2)$. Summing these payoffs for all the players gives $v(123) = 4$. This is the most the players can get if they form a grand coalition, and they can get this only if all the players cooperate. The central question in cooperative game theory is how to allocate the reward of 4 to the three players. In this example, player 2 contributes a payoff of -2 to the grand coalition, so should player 2 get an equal share of the 4? On the other hand, the 4 can only be obtained if player 2 agrees to play strategy B, so player 2 does have to be induced to do this. What would be a **fair allocation**?

One more observation is that player 3 seems to be in a bad position. On her own she can be guaranteed to get only $v(3) = -1$, but the assistance of player 1 does help since $v(13) = 1$. Player 2 doesn't do that well on her own but does do better with player 3.

Remark. There is a general formula for the characteristic function obtained by converting an n-person nonzero sum game to a cooperative game. Given any coalition $S \subset N$, the characteristic function is

$$v(S) = \max_{X \in X_S} \min_{Y \in Y_{N-S}} \sum_{i \in S} E_i(X, Y) = \min_{Y \in Y_{N-S}} \max_{X \in X_S} \sum_{i \in S} E_i(X, Y),$$

where X_S is the set of mixed strategies for the coalition S, Y_{N-S} is the set of mixed strategies for the coalition $N - S$, $E_i(X, Y)$ is the expected payoff to player $i \in S$, Since we now have equality throughout

$$v(S) + v(T) + v(N - (S \cup T)) = v(S \cup T) + v(N - (S \cup T)),$$

and so $v(S) + v(T) = v(S \cup T)$.

We need a basic definition regarding the allocation of rewards to each player. Recall that $v(N)$ represents the reward available if all players cooperate.

Definition 5.1.2 *Let x_i be a real number for each $i = 1, 2, \ldots, n$, with $\sum_i x_i \leq v(N)$. A vector $\vec{x} = (x_1, \ldots, x_n)$ is an* **imputation** *if*

- $x_i \geq v(i)$ **(individual rationality)**
- $\sum_{i=1}^{n} x_i = v(N)$ **(group rationality)**

Each x_i represents the share of the value of $v(N)$ received by player i. The imputation $\vec{x}$ is also called a **payoff vector** *or an* **allocation***, and we will use these words interchangeably.*

Remarks.

1. It is possible for x_i to be a negative number! That allows us to model coalition members that do not benefit and may be a detriment to a coalition.
2. Individual rationality means that the share received by player i should be at least what he could get on his own. Each player must be individually rational, or else why join the grand coalition?
3. Group rationality means any increase of reward to a player must be matched by a decrease in reward for one or more other players. Why is group rationality reasonable? Well, we know that $v(N) \geq \sum_i x_i \geq \sum_i v(i)$, just by definition. If in fact $\sum_i x_i < v(N)$, then each player could actually receive a bigger share than simply x_i; in fact, one possibility is an additional amount $(v(N) - \sum_i x_i)/n$. This says that the allocation x_i would be rejected by each player, so it must be true that $\sum_i x_i = v(N)$ for any reasonable allocation.
4. Any inessential game, i.e., $v(N) = \sum_{i=1}^{n} v(i)$, has one and only one imputation and it is $\vec{x} = (v(1), \ldots, v(n))$. The verification is a simple exercise (see the problems). These games are uninteresting because there is no incentive for any of the players to form any sort of coalition and there is no wiggle room in finding a better allocation.

The main objective in cooperative game theory is to determine the imputation that results in a **fair** allocation of the total rewards. Of course, this will depend on the definition of **fair**, as we mentioned earlier. That word is not at all precise. If you change the meaning of **fair** you will change the imputation.

We begin by presenting a way to transform a given characteristic function for a cooperative game to one which is frequently easier to work with. It is called the **(0,1) normalization of the original game**. This is not strictly necessary, but it does simplify the computations in many problems. The normalized game will result in a characteristic function with $v(i) = 0, v(N) = 1$. In addition, any two n-person cooperative games may be compared by comparing their normalized characteristic functions. If they are the same, the two games are said to be **strategically equivalent**.

The proof of the lemma will show how to make the conversion to $(0,1)$ normalized.

Lemma 5.1.3 *Any essential game with characteristic function v has a $(0,1)$ normalization with characteristic function v'; that is, given the characteristic function $v(\cdot)$ there is a unique characteristic function $v'(\cdot)$ that satisfies $v'(N) = 1, v'(i) = 0, 1 \leq i \leq n$, and $v'(S) = cv(S) + \sum_{i \in S} a_i$ for some constants $c > 0, a_1, \ldots, a_n$. The constants are given by*

$$c \equiv \frac{1}{v(N) - \sum_{i=1}^{n} v(i)} \quad \textit{and} \quad a_i \equiv -c\, v(i), \quad i = 1, 2, \ldots, n.$$

Proof. Consider the $n+1$ system of equations for constants $c, a_i, 1 \leq i \leq n$, given by

$$\begin{aligned} cv(i) + a_i &= 0, \quad 1 \leq i \leq n, \\ cv(N) + \sum_{i=1}^{n} a_i &= 1. \end{aligned}$$

If we add up the first n equations, we get $c \sum v(i) + \sum a_i = 0$. Subtracting this from the second equation results in

$$c[v(N) - \sum v(i)] = 1,$$

and we can solve for $c > 0$ because the game is essential (so $v(N) > \sum v(i)$). Now that we have c, we set $a_i = -c\, v(i)$. Solving this system, we get

$$c = \frac{1}{v(N) - \sum v(i)} > 0, \qquad a_i = -\frac{v(i)}{v(N) - \sum_i v(i)}.$$

Then, for any coalition $S \subset N$ define the characteristic function

$$v'(S) = cv(S) + \sum_{i \in S} a_i \quad \text{for all } S \subset N.$$

The equations we started with give immediately that $v'(N) = 1$ and $v'(i) = 0, i = 1, 2, \ldots, n.$ $\square$

So

$$a_1 = \frac{4}{15}, a_2 = -\frac{1}{15}, \text{ and } a_3 = \frac{4}{15}.$$

Then the normalized characteristic function by v' is calculated as

$$\begin{aligned} v'(i) &= \frac{4}{15}\, v(i) + a_i = 0 \\ v'(12) &= \frac{4}{15}\, v(12) + a_1 + a_2 = \frac{7}{15} \\ v'(13) &= \frac{4}{15}\, v(13) + a_1 + a_3 = \frac{4}{15} \\ v'(23) &= \frac{4}{15}\, v(23) + a_2 + a_3 = \frac{7}{15} \\ v'(123) &= \frac{4}{15}\, v(123) + a_1 + a_2 + a_3 = 1. \end{aligned}$$

In the rest of this section we let X denote the set of imputations $\vec{x}$. We look for an allocation $\vec{x} \in X$ as a **solution** to the game. The problem is the definition of the word **solution**. It is as vague as the word **fair**. We are seeking an imputation that, in some sense, is fair and allocates a fair share of the payoff to each player. To get a handle on the idea of **fair** we introduce the following subset of X.

Definition 5.1.4 *The* **reasonable allocation set** *of a cooperative game is a set of imputations $R \subset X$ given by*

$$R \equiv \{\vec{x} \in X \mid x_i \leq \max_{T \in \Pi^i} \{v(T) - v(T - i)\}, i = 1, 2, \ldots, n\},$$

where Π^i is the set of all coalitions for which player i is a member. So, if $T \in \Pi^i$, then $i \in T \subset N$, and $T - i$ denotes the coalition T without the player i.

In other words, the reasonable set is the set of imputations so that the amount allocated to each player is no greater than the maximum benefit that the player brings to any coalition of which the player is a member. The difference $v(T) - v(T - i)$ is the measure of the rewards for coalition T due to player i. The reasonable set gives us a first way to reduce the size of X and try to focus in on a solution.

If the reasonable set has only one element, which is extremely unlikely for most games, then that is our solution. If there are many elements in R, we need to cut it down further. In fact, we need to cut it down to the **core** imputations, or even further. Here is the definition.

Definition 5.1.5 *Let $S \subset N$ be a coalition and let $\vec{x} \in X$. The* **excess** *of coalition $S \subset N$ for imputation $\vec{x} \in X$ is defined by*

$$e(S, x) = v(S) - \sum_{i \in S} x_i.$$

PROBLEMS

5.1 Look back at Example 5.5. Find the normalized characteristic function and the normalized element in the least core.

5.2 Consider the bimatrix game with

$$A = \begin{bmatrix} 4 & 1 \\ 0 & 2 \end{bmatrix} \text{ and } B = \begin{bmatrix} 2 & 1 \\ 0 & 4 \end{bmatrix}.$$

(**a**) Find the characteristic function of this game.
(**b**) Find the core of the game $C(0)$.
(**c**) Find the least core.

5.3 Given the characteristic function $v(i) = 0, i = 1, 2, 3, 4$ and

$$v(12) = 4, v(13) = 4, v(14) = 3, v(23) = 6, v(24) = 2, v(34) = 2$$

$$v(123) = 10, v(124) = 7, v(134) = 7, v(234) = 8, v(1234) = 13,$$

find the normalized characteristic function. Given the fair allocation

$$\vec{x} = \left(\frac{1}{4}, \frac{33}{104}, \frac{33}{104}, \frac{3}{26}\right)$$

for the normalized game find the unnormalized allocation.

5.4 Find the characteristic function for the following three–player game. Each player has two strategies, A, B. If player 1 plays A the matrix is

$$\begin{bmatrix} (1,2,1) & (3,0,1) \\ (-1,6,-3) & (3,2,1) \end{bmatrix},$$

while if player 1 plays B the matrix is

$$\begin{bmatrix} (-1,2,4) & (1,0,3) \\ (7,5,4) & (3,2,1) \end{bmatrix}.$$

In each matrix player 2 is the row player and player 3 is the column player. Next find the normalized characteristic function.

5.5 Derive the least core for the game with

$$v(123) = 1 = v(12) = v(13) = v(23) \quad \text{and} \quad v(1) = v(2) = v(3) = 0.$$

5.6 Given the characteristic function

$$v(1) = 1, v(2) = \frac{1}{4}, v(3) = -1, v(12) = 3, v(13) = -1, v(23) = 1, v(123) = 4,$$

find the least core without normalizing.

5.7 A **constant sum game** is one in which $v(S) + v(N - S) = v(N)$ for all coalitions $S \subset N$. Show that any essential constant sum game must have empty core $C(0) = \emptyset$.

5.8 In this problem you will see why inessential games are of no interest. Show that an **inessential** game has one and only one imputation and is given by

$$\vec{x} = (x_1, \ldots, x_n) = (v(1), v(2), \ldots, v(n));$$

that is, each player is allocated exactly the benefit of the one-player coalition.

5.9 A player i is a **dummy** if $v(S) = v(S \cup i)$, for every $S \subset N$. It looks like a dummy contributes nothing. Show that if i is a dummy, $v(i) = 0$, and if $\vec{x} \in C(0)$, then $x_i = 0$.

5.10 Show that a vector $\vec{x} = (x_1, x_2, \ldots, x_n)$ is an imputation if and only if there are nonnegative constants $a_i \geq 0, i = 1, 2, \ldots, n$, such that $\sum_{i=1}^n a_i = v(N) - \sum_{i=1}^n v(i)$, and $x_i = v(i) + a_i$ for each $i = 1, 2, \ldots, n$.

5.11 Let $\delta_i = v(N) - v(N - i)$. Show that $C(0) = \emptyset$ if $\sum_{i=1}^n \delta_i < v(N)$.

5.12 Verify the statement: $C(0) \neq \emptyset$ if and only if the linear program

$$\text{Minimize } z = x_1 + \cdots + x_n$$
$$\text{subject to } v(S) \leq \sum_{i \in S} x_i \text{ for every } S \subsetneq N$$

has a finite minimum, say z^*, and $z^* \leq v(N)$.

5.1.1 Finding the Least Core

The next theorem formalizes the idea above that when $e(S, \vec{x}) \leq 0$ for all coalitions, then the player should be happy with the imputation $\vec{x}$ and would not want to switch to another one.

One way to describe the fact that one imputation is better than another is the concept of domination.

Definition 5.1.8 *If we have two imputations $\vec{x} \in X, \vec{y} \in X$, and a nonempty coalition $S \subset N$, then $\vec{x}$* **dominates** *$\vec{y}$ (for the coalition S) if $x_i > y_i$ for all members $i \in S$, and $\vec{x}(S) = \sum_{i \in S} x_i \leq v(S)$.*

If $\vec{x}$ dominates $\vec{y}$ for the coalition S, then members of S prefer the allocation $\vec{x}$ to the allocation $\vec{y}$, because they get more $x_i > y_i$, for each $i \in S$, and the coalition S can

EXAMPLE 5.9

This example[1] will present a game with an empty core. We will see that when we calculate the least core $X^1 = C(\varepsilon^1)$, where ε^1 is the smallest value for which $C(\varepsilon^1) \neq \emptyset$, we will obtain a reasonable fair allocation (and hopefully only one). The fact that $C(0) = \emptyset$ means that when we calculate ε^1 it must be the case that $\varepsilon^1 > 0$ because if $\varepsilon^1 < 0$, by the definition of ε^1 as the smallest ε making $C(\varepsilon) \neq \emptyset$, we know immediately that $C(0) \neq \emptyset$ because $C(\varepsilon)$ increases as ε gets bigger.

Suppose that Bill has 150 sinks to give away to whomever shows up to take them away. Amy(1), Agnes(2), and Agatha(3) simultaneously show up with their trucks to take as many of the sinks as their trucks can haul. Amy can haul 45, Agnes 60, and Agatha 75, for a total of 180, 30 more than the maximum of 150. The wrinkle in this problem is that the sinks are too heavy for any one person to load onto the trucks so they must cooperate in loading the sinks. The question is: How many sinks should be allocated to each person?

Define the characteristic function $v(S)$ as the number of sinks the coalition $S \subset N = \{1, 2, 3\}$ can load. We have $v(i) = 0, i = 1, 2, 3$, since they must cooperate to receive any sinks at all, and

$$v(12) = 105,\ v(13) = 120,\ v(23) = 135,\ v(123) = 150.$$

It will be easier to not normalize this problem, so the set of imputations will be $X = \{(x_1, x_2, x_3) | x_i \geq 0, \sum x_i = 150\}$. First let's use Maple to see if the core is nonempty:

```
> with(simplex):
> obj:=x1+x2+x3:
> cnsts:={105-x1-x2<=0,120-x1-x3<=0,135-x2-x3<=0};
> minimize(obj,cnsts,NONNEGATIVE);
> assign(%);
> obj;
```

Maple gives the output $x_1 = 45, x_2 = 60, x_3 = 75$, and $obj = x_1 + x_2 + x_3 = 180 > v(123) = 150$. So the core of this game is empty. A direct way to get this is to note that the inequalities

$$x_1 + x_2 \geq 105, x_1 + x_3 \geq 120 \text{ and } x_2 + x_3 \geq 135$$

imply that $2(x_1 + x_2 + x_3) = 2(150) = 300 \geq 360$, which is impossible.

[1] due to Mesterton-Gibbons [15].

The next step is to calculate the least core. Begin with the definition:

$$\begin{aligned}C(\varepsilon) &= \{\vec{x} \in X \mid e(S,\vec{x}) \le \varepsilon, \forall S \subsetneq N\}\\ &= \{\vec{x} \in X \mid v(S) - \sum_{i \in S} x_i \le \varepsilon\}\\ &= \{\vec{x} \mid 105 \le x_1 + x_2 + \varepsilon, 120 \le x_1 + x_3 + \varepsilon,\\ &\qquad 135 \le x_2 + x_3 + \varepsilon, -x_i \le \varepsilon\}.\end{aligned}$$

We know that $x_1 + x_2 + x_3 = 150$ so by replacing $x_3 = 150 - x_1 - x_2$ we obtain as conditions on ε that

$$120 \le 150 - x_2 + \varepsilon,\ 135 \le 150 - x_1 + \varepsilon,\ 105 \le x_1 + x_2 + \varepsilon.$$

We see that $45 \ge x_1 + x_2 - 2\varepsilon \ge 105 - 3\varepsilon$, implying that $\varepsilon \ge 20$. This is in fact the smallest $\varepsilon^1 = 20$, making $C(\varepsilon) \ne \emptyset$. Using $\varepsilon^1 = 20$, we calculate

$$C(20) = \{(x_1 = 35, x_2 = 50, x_3 = 65)\}.$$

Hence the fair allocation is to let Amy have 35 sinks, Agnes 50, and Agatha 65 sinks, and they all cooperate.

We conclude that our fair allocation of sinks is as follows:

player	Truck capacity	Allocation
Amy	45	35
Agnes	60	50
Agatha	75	65
Total	180	150

Observe that each player in the fair allocation gets 10 less than the capacity of her truck. It seems that this is certainly a reasonably fair way to allocate the sinks; that is, there is an undersupply of 30 sinks so each player will receive $\frac{30}{3} = 10$ less than her truck can haul. You might think of other ways in which you would allocate the sinks (e.g, maybe it would be better to fill the large trucks first), but the solution here minimizes the maximum dissatisfaction over any other allocation for all coalitions.

The least core plays a critical role in solving the problem when $C(0) = \emptyset$ or there is more than one allocation in $C(0)$.

Lemma 5.1.10 *Let*

$$\varepsilon^1 = \min_{\vec{x} \in X} \max_{S \subsetneq N} e(S,\vec{x}).$$

Then the least core $X^1 = C(\varepsilon^1) \ne \emptyset$ *and if* $\varepsilon > \varepsilon^1$, *then* $C(\varepsilon^1) \subsetneq C(\varepsilon)$.

Proof. Since the set of imputations is compact (=closed and bounded) and $\vec{x} \mapsto \max_S e(S, \vec{x})$ is at least lower semicontinuous, there is an allocation $\vec{x}_0$ so that the minimum in the definition of ε^1 is achieved, namely, $\varepsilon^1 = \max_S e(S, \vec{x}_0) \geq e(S, \vec{x}_0), \forall S \subsetneq N$. This is the very definition of $\vec{x}_0 \in C(\varepsilon^1)$ and so $C(\varepsilon^1) \neq \emptyset$.

On the other hand, if we have a smaller $\varepsilon < \varepsilon^1 = \min_{\vec{x}} \max_{S \subsetneq N}$, then for every allocation $\vec{x} \in X$, we have $\varepsilon < \max_S e(S, \vec{x})$. So, for any allocation there is at least one coalition $S \subsetneq N$ for which $\varepsilon < e(S, \vec{x})$. This means that for this ε, no matter which allocation is given, $\vec{x} \notin C(\varepsilon)$. Thus, $C(\varepsilon) = \emptyset$. As a result, ε^1 is the smallest ε so that $C(\varepsilon) \neq \emptyset$. $\square$

Remarks.

These remarks summarize the ideas behind the use of the least core.

1. For a given grand allocation $\vec{x}$, the coalition S_0 that most objects to $\vec{x}$ is the coalition giving the largest excess and so satisfies

$$e(S_0, \vec{x}) = \max_{S \subsetneq N} e(S, \vec{x}).$$

For each fixed coalition S, the allocation giving the minimum dissatisfaction is

$$e(S, \vec{x}_0) = \min_{\vec{x} \in X} e(S, \vec{x}).$$

2. The value of ε giving the least ε-core is

$$\varepsilon^1 \equiv \min_{\vec{x} \in X} \max_{S \subsetneq N} e(S, \vec{x}),$$

and this is the smallest level of dissatisfaction.

3. If $\varepsilon^1 = \min_{\vec{x}} \max_{S \subsetneq N} e(S, \vec{x}) < 0$, then there is at least one allocation $\vec{x}^*$ that satisfies $\max_S e(S, \vec{x}^*) < 0$. That means that $e(S, \vec{x}^*) < 0$ for every coalition $S \subsetneq N$. Every coalition is satisfied with $\vec{x}^*$ because $v(S) < \vec{x}^*(S)$, so that every coalition is allocated at least its maximum value.

If $\varepsilon^1 = \min_{\vec{x}} \max_{S \subsetneq N} e(S, \vec{x}) > 0$, then for every allocation $\vec{x} \in X, \max_S e(S, \vec{x}) > 0$. Consequently, there is at least one coalition S so that $e(S, \vec{x}) = v(S) - \vec{x}(S) > 0$. For any allocation, there is at least one coalition that will not be happy with it.

4. The excess function $e(S, \vec{x})$ is a measure of dissatisfaction of S with the imputation $\vec{x}$. It makes sense that the best imputation would minimize the largest dissatisfaction over all the coalitions. This leads us to one possible definition of a solution for the n-person cooperative game. An allocation $\vec{x}^* \in X$ is a solution to the cooperative game if

$$\varepsilon^1 = \min_{\vec{x} \in X} \max_S e(S, \vec{x}) = \max_S e(S, \vec{x}^*),$$

so that $\vec{x}^*$ minimizes the maximum excess for any coalition S. When there is only one such allocation $\vec{x}^*$, it is the fair allocation. The problem is that there may be more than one element in the least core, then we still have a problem as to how to choose among them.

Remark: Maple Calculation of the Least Core. The point of calculating the ε-core is that the core is not a sufficient set to ultimately solve the problem in the case when the core $C(0)$ is (1) empty or (2) consists of more than one point. In case (2) the issue, of course, is which point should be chosen as the fair allocation. The ε-core seeks to address this issue by shrinking the core at the same rate from each side of the boundary until we reach a single point. We can use Maple to do this.

The calculation of the least core is equivalent to the linear programming problem

$$\begin{aligned}&\text{Minimize } z\\&\text{subject to}\\&v(S) - \vec{x}(S) = v(S) - \sum_{i \in S} x_i \leq z, \text{ for all } S \subsetneq N.\end{aligned}$$

The characteristic function need not be normalized. So all we really need to do is to formulate the game using characteristic functions, write down the constraints, and plug them into Maple. The result will be the smallest $z = \varepsilon^1$ that makes $C(\varepsilon^1) \neq \emptyset$, as well as an imputation which provides the minimum.

For example, let's suppose we start with the characteristic function

$$v(i) = 0,\ i = 1,2,3,\ v(12) = 2, v(23) = 1, v(13) = 0,\ \ v(123) = \frac{5}{2}.$$

The constraint set is the ε-core

$$\begin{aligned}C(\varepsilon) &= \{\vec{x} = (x_1, x_2, x_3) \mid v(S) - x(S) \leq \varepsilon, \emptyset \neq S \subsetneq N\}\\&= \{-x_i \leq \varepsilon, i = 1,2,3, 2 - x_1 - x_2 \leq \varepsilon, 1 - x_2 - x_3 \leq \varepsilon,\\&\qquad\qquad 0 - x_1 - x_3 \leq \varepsilon, x_1 + x_2 + x_3 = \frac{5}{2}\}\end{aligned}$$

The Maple commands used to solve this are very simple:

```
> with(simplex):
> cnsts:={-x1<=z,-x2<=z,-x3<=z,2-x1-x2<=z,1-x2-x3<=z,-x1-x3<=z,
          x1+x2+x3=5/2};
> minimize(z,cnsts);
```

Maple produces the output

$$x_1 = \frac{5}{4},\ x_2 = 1,\ x_3 = \frac{1}{4},\ z = -\frac{1}{4}.$$

property (one of the four). If n bags of garbage are dumped on a coalition S of property owners, the coalition receives a reward of $-n$. The characteristic function is taken to be the best that the members of a coalition S can do, which is to dump all their garbage on the property of the owners not in S.

(a) Explain why the characteristic function should be $v(S) = -(4 - |S|)$, where $|S|$ is the number of members in S.

(b) Show that the core of the game is empty.

(c) Recall that an imputation $\vec{y}$ dominates an imputation $\vec{x}$ through the coalition S if $e(S, \vec{y}) \geq 0$ and $y_i > x_i$ for each component i. Find a coalition S so that $\vec{y} = (-1.5, -0.5, -1, -1)$ dominates $\vec{x} = (-2, -1, -1, 0)$.

5.2 THE NUCLEOLUS

The core $C(0)$ might be empty, but we can find an ε so that $C(\varepsilon)$ is not empty. We can fix the **empty** problem. Even if $C(0)$ is not empty, it may contain more than one point and again we can use $C(\varepsilon)$ to **maybe** shrink the core down to one point or, if $C(0) = \emptyset$, to expand the core until we get it nonempty. The problem is what happens when the least core $C(\varepsilon)$ itself has too many points.

In this section we will address the issue of what to do when the least core $C(\varepsilon)$ contains more than one point. Remember that $e(S, \vec{x}) = v(S) - \sum_{i \in S} x_i = v(S) - \vec{x}(S)$ and the larger the excess, the more unhappy the coalition S is with the allocation $\vec{x}$. So, no matter what, we want the excess to be as small as possible for all coalitions and we want the imputation which achieves that.

In the previous section we saw that we should shrink $C(0)$ to $C(\varepsilon^1)$, so if $C(\varepsilon^1)$ has more than one allocation, why not shrink that also? No reason at all.

Let's begin by working through an example to see how to shrink the ε^1-core.

■ **EXAMPLE 5.10**

Let us take the normalized characteristic function for the three–player game

$$v(12) = \frac{4}{5}, v(13) = \frac{2}{5}, v(23) = \frac{1}{5} \text{ and } v(123) = 1, v(i) = 0, \quad i = 1, 2, 3.$$

Step 1: Calculate the least core. We have the ε-core

$$\begin{aligned} C(\varepsilon) &= \{(x_1, x_2, x_3) \in X \mid e(S, x) \leq \varepsilon, \forall\, \emptyset \neq S \subsetneq N\} \\ &= \{(x_1, x_2, 1 - x_1 - x_2) \mid -\varepsilon \leq x_1 \leq \frac{4}{5} + \varepsilon, \\ &\qquad -\varepsilon \leq x_2 \leq \frac{3}{5} + \varepsilon, \ \ \frac{4}{5} - \varepsilon \leq x_1 + x_2 \leq 1 + \varepsilon\}. \end{aligned}$$

and X^2 consists of exactly one point. That is our solution to the problem. Notice that for this allocation

$$e(13,\vec{x}) = x_2 - \frac{3}{5} = \frac{7}{20} - 12/20 = -\frac{1}{4}$$
$$e(23,\vec{x}) = x_1 - \frac{4}{5} = 11/20 - \frac{4}{5} = -\frac{1}{4}$$
$$e(1,\vec{x}) = -11/20, \quad \text{and} \quad e(2,x) = -\frac{7}{20},$$

and each of these is a constant smaller than $-\frac{1}{10}$. Because they are all independent of any specific allocation, we know that they cannot be reduced any further by adjusting the imputation. Since X^2 contains only one allocation, no further adjustments are possible in any case. This is the allocation that minimizes the maximum dissatisfaction of all coalitions.

The most difficult part of this procedure is finding ε^1, ε^2, and so on. This is where Maple is a great help. For instance, we can find $\varepsilon^2 = -\frac{1}{4}$ very easily if we use the commands

```
> with(simplex):
> cnsts:={-x1<=z,-x2<=z,x2-3/5<=z,x1-4/5<=z,x1+x2=9/10};
> minimize(z,cnsts);
```

Maple informs us that `z=-1/4,x1=11/20,x2=7/20`, but you have to be careful of the `x1` and `x2` because these are points providing the minimum but you don't know whether they are the only such points. That is what you must verify.

In general, we would need to continue this procedure if X^2 also contained more than one point. Here are the sequence of steps to take in general until we get down to one point:

1. **Step 0: Initialize.** We start with the set of all possible imputations X and the coalitions excluding N and $\emptyset$:

$$X^0 \equiv X, \quad \Sigma^0 \equiv \{S \subsetneq N, S \neq \emptyset\}$$

2. **Step $k \geq 1$: Successively calculate**

   (**a**) The minimum of the maximum dissatisfaction

$$\varepsilon^k \equiv \min_{\vec{x} \in X^{k-1}} \max_{S \in \Sigma^{k-1}} e(S,\vec{x}).$$

   (**b**) The set of allocations achieving the minimax dissatisfaction

$$X^k \equiv \{\vec{x} \in X^{k-1} \mid \varepsilon^k = \min_{\vec{x} \in X^{k-1}} \max_{S \in \Sigma^{k-1}} e(S,\vec{x}) = \max_{S \in \Sigma^{k-1}} e(S,\vec{x})\}$$
$$= \{\vec{x} \in X^{k-1} \mid e(S,\vec{x}) \leq \varepsilon^k, \ \forall S \subsetneq \Sigma^{k-1}\}.$$

To get rid of the first equality so that we can continue, use

```
> gcnsts:=fcnsts[2..7];
```

This puts the second through seventh elements of `fcnsts` into `gcnsts`. Now, to see if there are other solutions, we need to solve the system of inequalities in `gcnsts` for x_1, x_2. Maple does that as follows:

```
> with(SolveTools:-Inequality):
> glc:=LinearMultivariateSystem(gcnsts,[x2,x3]);
```

Maple solves the system of inequalities in the sense that it reduces the inequalities to simplest form and gives the following output:

```
{x2 <=3/4,1/3<=x2} {x3<=-x2+11/12, -x2+11/12<=x3}.
```

We see that $\frac{1}{3} \leq x_2 \leq \frac{3}{4},\ x_2 + x_3 = \frac{11}{12}$ and $x_1 = \frac{1}{12}$.

2. To get the second linear program we first have to see which coalitions are dropped. First we assign the variables that are known from the first linear program and recalculate the constraints:

```
> assign(x1=1/12,z=-1/12);
> cnsts1:={v1-x1<=z,v2-x2<=z,v3-x3<=z,v12-(x1+x2)<=z,
           v13-(x1+x3)<=z,v23-(x2+x3)<=z,x1+x2+x3=v123};
```

Maple gives the output:

```
cnsts1:={-x3 <= -1/12, -x2-x3<=-11/12,-x2 <=-1/12,-x2 <= -1/3,
 -1/12 <= -1/12,-x3 <=-1/6, 1/12+x2+x3=1}.
```

Getting rid of the coalitions that have excess$= -\frac{1}{12}$ (indicated by the output without any x variables), we have the new constraint set

```
> cnsts2:={v2-x2<=z2,v3-x3<=z2,v12-(x1+x2)<=z2,
             v13-(x1+x3)<=z2,x1+x2+x3=v123};
```

Now we solve the second linear program

```
> minimize(z2,cnsts2);
```

which gives

$$x_2 = \frac{13}{24}, x_3 = \frac{3}{8}, z_2 = -\frac{7}{24}.$$

At each stage we need to determine whether there is more than one solution of the linear programming problem. To do that, we have to substitute our solution for z_2 into the constraints and solve the inequalities:

```
> fcnsts2:=subs(z2=-7/24, 11/12=x2+x3,cnsts2);
> gcnsts2:=fcnsts2[2..5] union
                  {x2+x3<=11/12,x2+x3>=11/12};
> g1c2:=LinearMultivariateSystem(gcnsts2,[x2,x3]);
```

We get

```
glc2:={x2=13/24,x3 <= x2+x3=11/12},
```

and we know now that $x_1 = \frac{1}{12}, x_2 = \frac{13}{24}$, and $x_3 = \frac{3}{8}$ because $x_2 + x_3 = \frac{11}{12}$.

We could continue setting up linear programs until we get the set of empty coalitions, but there is no point to that when we are doing it by hand (or with a Maple assist), because we are now at the point when we have one and only one allocation.

So we are finally done, and we conclude that

$$\text{Nucleolus} = X^2 = \left\{ \left(\frac{1}{12}, \frac{13}{24}, \frac{3}{8} \right) \right\}.$$

The first $\varepsilon^1 = -\frac{1}{12}$ and the second $\varepsilon^2 = -\frac{7}{24}$.

Now for a another practical application.

EXAMPLE 5.13

Three cities are to be connected to a water tower at a central location. Label the three cities $1, 2, 3$ and the water tower as 0. The cost to lay pipe connecting location i with location j is denoted as $c_{ij}, i \neq j$. Figure 5.6 contains the data for our problem.

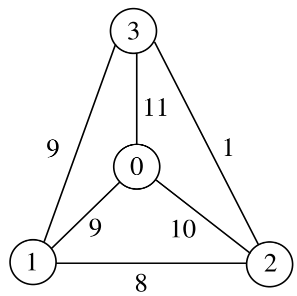


Figure 5.6 Three cities and a water tower.

Coalitions among cities can form for pipe to be laid to the water tower. For example, it is possible for city 1 and city 3 to join up so that the cost to the coalition $\{13\}$ would be the sum of the cost of going from 1 to 3 and then 3 to 0. It may be possible to connect from 1 to 3 to 0 but not from 3 to 1 to 0 depending on land conditions. We have the following costs in which we do not treat the water tower as a player:

$$c_1 = 9, c_2 = 10, c_3 = 11, c_{123} = 18,$$
$$c_{12} = 17, c_{13} = 18, c_{23} = 11.$$

The single-player coalitions correspond to hooking up that city directly to location 0. Converting this to a savings game, we let $c(S)$ be the total cost for formation of the grand coalition, we have

$$\underbrace{(1)(2)\cdots(|S|-2)(|S|-1)}_{|S|-1 \text{ arrive}} \quad \underbrace{(i)}_{i \text{ arrives}} \quad \underbrace{(n-|S|)(n-|S|-1)\cdots(2)(1)}_{\text{remaining arrive}}$$

Remember that because a characteristic function is superadditive, the players have the incentive to form the grand coalition.

For a given coalition S, by elementary probability, there are $(|S|-1)!(n-|S|)!$ ways i can join the grand coalition N, joining S first. With this reasoning, we assume that Z_i has the probability distribution

$$Prob(Z_i = S) = \frac{(|S|-1)!(n-|S|)!}{n!}.$$

We choose this distribution because $|S|-1$ players have joined before player i, and this can happen in $(|S|-1)!$ ways; and $n-|S|$ players join after player i, and this can happen in $(n-|S|)!$ ways. The denominator is the total number of ways that the grand coalition can form among n players. Any of the $n!$ permutations has probability $\frac{1}{n!}$ of actually being the way the players join. This distribution assumes that they are **all equally likely**. One could debate this choice of distribution, but this one certainly seems reasonable. Also, see Example 5.17 below for a direct example of the calculation of the arrival of a player to a coalition and the consequent benefits.

Therefore, for the fixed player i, the benefit player i brings to the coalition Z_i is $v(Z_i) - v(Z_i - i)$. It seems reasonable that the amount of the total grand coalition benefits that should be allocated to player i should be the expected value of $v(Z_i) - v(Z_i - i)$. This gives,

$$\begin{aligned} x_i \equiv E[v(Z_i) - v(Z_i - i)] &= \sum_{\{S \in \Pi_i\}} [v(S) - v(S-i)] Prob(Z_i = S) \\ &= \sum_{\{S \in \Pi^i\}} [v(S) - v(S-i)] \frac{(|S|-1)!(n-|S|)!}{n!}. \end{aligned}$$

The **Shapley value (or vector)** is then the allocation $\vec{x} = (x_1, \ldots, x_n)$. At the end of this chapter you can find the Maple code to find the Shapley value.

■ EXAMPLE 5.14

Two players have to divide \$M, but they each get zero if they can't reach an agreement as to how to divde it. What is the fair division? Obviously, without regard to the benefit derived from the money the allocation should be $M/2$ to each player. Let's see if Shapley gives that.

Define $v(1) = v(2) = 0, v(12) = M$. Then

$$x_1 = [v(1) - v(\emptyset)] \frac{0!1!}{1!} 2! + [v(12) - v(2)] \frac{1!0!}{2!} = \frac{M}{2}.$$

For example, $v(134) = \max(u_1 + u_3 + u_4 - 100, 0) = 125 - 100 = 25$. We compute

$$\begin{aligned} x_1 &= \sum_{\{S \mid 1 \in S, v(S-1)>0\}} u_1 \frac{(|S|-1)!(4-|S|)!}{4!} \\ &\quad + \sum_{\{S \mid 1 \in S, v(S)>0, v(S-1)=0\}} \left(\sum_{i \in S} u_i - T \right) \frac{(|S|-1)!(4-|S|)!}{4!} \\ &= \frac{2!1!}{4!} \cdot u_1 + \frac{3!0!}{4!} u_1 \\ &\quad + \frac{1!2!}{4!} ([u_1 + u_4 - 100]) + \frac{2!1!}{4!} [u_1 + u_3 + u_4 - 100] \\ &= \frac{65}{6}. \end{aligned}$$

The first term comes from coalition $S = 124$; the second term, from coalition $S = 1234$; the third term comes from coalition $S = 14$; and the last term from coalition $S = 134$.

As a result, the amount player 1 will be billed will be $z_1 = u_1 - x_1 = 25 - \frac{65}{6} = \frac{85}{6}$ thousand dollars. In a similar way we calculate

$$x_2 = \frac{40}{3}, \quad x_3 = \frac{25}{3}, \quad \text{and} \quad x_4 = \frac{45}{2},$$

so that the actual bill to each player will be

$$\begin{aligned} z_1 &= 25 - \tfrac{65}{6} = 14.167, \\ z_2 &= 30 - \tfrac{40}{3} = 16.667, \\ z_3 &= 20 - \tfrac{25}{3} = 11.667, \\ z_4 &= 80 - \tfrac{45}{2} = 57.5. \end{aligned}$$

For comparison purposes it is not too difficult to calculate the nucleolus for this game to be $\left(\frac{25}{2}, 15, 10, \frac{35}{2}\right)$, so that the payments using the nucleolus will be

$$\begin{aligned} z_1 &= 25 - \tfrac{25}{2} = \tfrac{25}{2} = 12.5, \\ z_2 &= 30 - 15 = 15, \\ z_3 &= 20 - 10 = 10, \\ z_4 &= 80 - \tfrac{35}{2} = \tfrac{125}{2} = 62.5. \end{aligned}$$

There is yet a third solution, the straightforward solution that assesses the amount to each player in proportion to each household's maximum payment to the total assessment. For example, $u_1/(\sum_i u_i) = 25/155 = 0.1613$ and so player 1 could be assessed the amount $0.1613 \times 100 = 16.13$.

The horizontal axis (abscissa) is the payoff to player I, and the vertical axis (ordinate) is the payoff to player II. Any point in the parabolic region is achievable for some $0 \le x \le 1, 0 \le y \le 1$.

The parabola is given by the implicit equation $5(E_1 - E_2)^2 - 2(E_1 + E_2) + 1 = 0$. If the players play pure strategies, the payoff to each player will be at one of the vertices. The pure Nash equilibria yield the payoff pairs $(E_1 = 1, E_2 = 2)$ and $(E_1 = 2, E_2 = 1)$. The mixed Nash point gives the payoff pair $(E_1 = \frac{1}{5}, E_2 = \frac{1}{5})$, which is strictly inside the region of points, called the **noncooperative payoff set.**

Now, if the players do not cooperate, they will achieve one of two possibilities: (1) The vertices of the figure, if they play pure strategies; or (2) any point in the region of points bounded by the two lines and the parabola, if they play mixed strategies. The portion of the triangle outside the parabolic region is **not** achievable simply by the players using mixed strategies. However, if the players agree to cooperate, then any point on the boundary of the triangle, the entire shaded region,[7] including the boundary of the region, are achievable payoffs, which we will see shortly. Cooperation here means an agreement as to which combination of strategies each player will use and the proportion of time that the strategies will be used.

Player I wants a payoff as large as possible and thus as far to the right on the triangle as possible. Player II wants to go as high on the triangle as possible. So player I wants to get the payoff at $(2, 1)$, and player II wants the payoff at $(1, 2)$, but this is possible if and only if the opposing player agrees to play the correct strategy. In addition, it seems that nobody wants to play the mixed Nash equilibrium because they can both do better, but they have to cooperate to achieve a higher payoff.

Here is another example illustrating the achievable payoffs.

EXAMPLE 5.21

	II_1	II_2	II_3
I_1	$(1, 4)$	$(-2, 1)$	$(1, 2)$
I_2	$(0, -2)$	$(3, 1)$	$(\frac{1}{2}, \frac{1}{2})$

We will draw the pure payoff points of the game as the vertices of the graph and connect the pure payoffs with straight lines, as in Figure 5.9. The vertices of the polygon are the payoffs from the matrix. The solid lines connect the pure payoffs. The dotted lines extend the region of payoffs to those payoffs that **could** be achieved if both players cooperate. For example, suppose that player I always chooses row 2, I_2, and player II plays the mixed strategy $Y = (y_1, y_2, y_3)$, where $y_i \ge 0, y_1 + y_2 + y_3 = 1$. The expected payoff to I is then

$$E_1(2, Y) = 0\; y_1 + 3y_2 + \frac{1}{2}\; y_3,$$

[7] This region is called the **convex hull** of the pure payoff pairs. The convex hull of a set of points is the smallest convex set containing all the points.

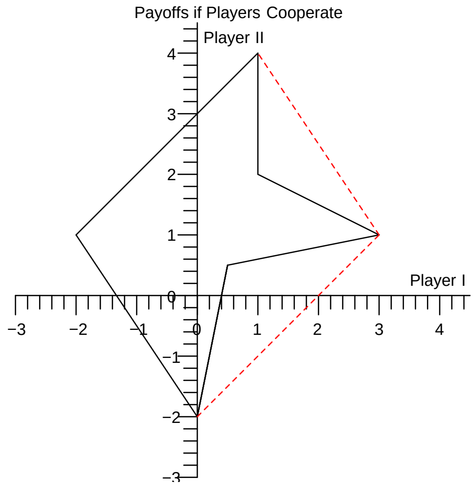


Figure 5.9 Achievable payoffs with cooperation.

and the expected payoff to II is

$$E_2(2, Y) = -2y_1 + 1y_2 + \frac{1}{2}\, y_3.$$

Hence

$$(E_1, E_2) = y_1(0, -2) + y_2(3, 1) + y_3\left(\frac{1}{2}, \frac{1}{2}\right),$$

which, as a linear combination of the three points $(0, -2)$, $(3, 1)$, and $(\frac{1}{2}, \frac{1}{2})$, is in the convex hull of these three points. This means that if players I and II can agree that player I will always play row 2, then player II can choose a (y_1, y_2, y_3) so that the payoff pair to each player will be in the triangle bounded by the lower dotted line in Figure 5.9 and the lines connecting $(0, -2)$ with $(\frac{1}{2}, \frac{1}{2})$ with $(3, 1)$. The conclusion is that any point in the convex hull of all the payoff points is achievable if the players agree to cooperate.

One thing to be mindful of is that the Figure 5.9 does not show the actual payoff pairs that are achievable in the noncooperative game as we did for the 2×2 prisoner's dilemma game (Figure 5.8) because it is too involved. The boundaries of that region may not be straight lines or parabolas.

The entire 4-sided region in Figure 5.9 is called the **feasible set** for the problem. The precise definition in general is as follows.

Definition 5.4.1 *The* **feasible set** *is the convex hull of all the payoff points corresponding to pure strategies of the players.*

know the line where the maximum occurs, which here is $v = -2u+5$, because then we may substitute into g and use calculus:

$$f(u) = g(u, -2u+5) = (u + \frac{1}{4})(-2u + \frac{16}{3})$$
$$\implies f'(u) = -4u + \frac{29}{6} = 0$$
$$\implies u = \frac{29}{24}.$$

So this gives us the solution as well.

4. **Find the strategies giving the negotiated solution.** How should the players cooperate in order to achieve the bargained solutions we just obtained? To find out, the only points in the bimatrix that are of interest are the endpoints of the Pareto-optimal boundary, namely, $(1,3)$ and $(2,1)$. So the cooperation must be a linear combination of the strategies yielding these payoffs. Solve

$$\left(\frac{29}{24}, \frac{31}{12}\right) = \lambda(1,3) + (1-\lambda)(2,1),$$

to get $\lambda = \frac{19}{24}$. This says that (I,II) must agree to play (row 1,col 1) with probability $\frac{19}{24}$ and (row 2, col 2) with probability $\frac{5}{24}$.

The Nash bargaining theorem also applies to games in which the players have payoff functions $u_1(x,y), u_2(x,y)$, where x, y are in some interval and the players have a continuum of strategies. As long as the feasible set contains some security point u_1^*, u_2^*, we may apply Nash's theorem. Here is an example.

■ **EXAMPLE 5.26**

Suppose that two persons are given $1000, which they can split if they can agree on how to split it. If they cannot agree they each get nothing. One player is rich, so her payoff function is

$$u_1(x) = \frac{x}{2}, \quad 0 \le x \le 1000$$

because the receipt of more money will not mean that much. The other player is poor, so his payoff function is

$$u_2(y) = \ln(y+1), \quad 0 \le y \le 1000,$$

because small amounts of money mean a lot but the money has less and less impact as he gets more but no more than $1000. We want to find the bargained solution. The safety points are taken as $(0,0)$ because that is what they get

(u^t, v^t), whatever the point is, has the equation

$$v - v^t = -m_p(u - u^t).$$

It is proved in [7] that this line *must* pass through the optimal threat security point.

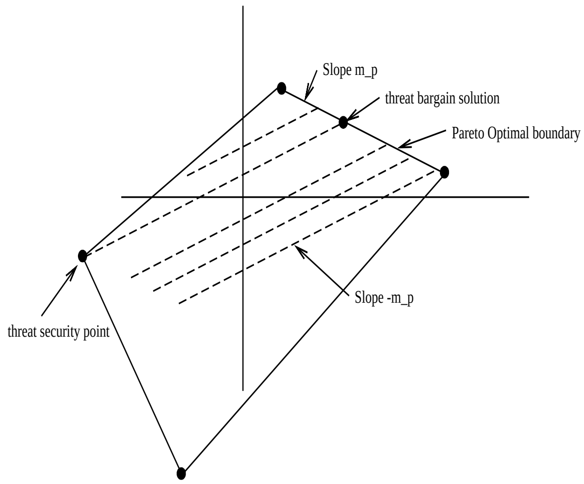


Figure 5.18 Lines through possible threat security points.

The equation of the Pareto-optimal boundary line is

$$v = m_p u + b = -\frac{3}{8}u + \frac{38}{8},$$

so the intersection point of the two lines will be at the coordinates

$$\overline{u} = \frac{m_p\ u^t + v^t - b}{2m_p} = \frac{3u^t - 8v^t + 38}{6},$$
$$\overline{v} = \frac{1}{2}(m_p u^t + v^t + b) = \frac{-(3u^t - 8v^t) + 38}{16}.$$

Now, remember that we are trying to find the best threat strategies to use, but the primary objective of the players is to maximize their payoffs $\overline{u}, \overline{v}$. This tells us exactly what to do to find the optimal threat security point.

- Player I will **maximize** $\overline{u}$ if she chooses threat strategies to maximize the quantity $-m_p\ u^t - v^t = \frac{3}{8}\ u^t - v^t$.
- Player II will maximize $\overline{v}$ if he chooses threat strategies to **minimize** the same quantity $-m_p\ u^t - v^t$ because the Pareto-optimal boundary will have $m_p < 0$, so the sign of the term multiplying u^t will be opposite in $\overline{u}$ and $\overline{v}$.

at another job. In negotiations with the union, the firm agrees to the pay level p and to employ $0 \le w \le W$ workers. We may consider the payoff functions as

$$u(p, w) = f(w) - pw \quad \text{to the company}$$

and

$$v(p, w) = pw + (W - w)p_0 \quad \text{to the union.}$$

Assume the safety security point is $u^* = 0$ for the company and $v^* = Wp_0$ for the union.

(**a**) What is the nonlinear program to find the Nash bargaining solution?

(**b**) Assuming an interior solution, show that the solution (p^*, w^*) of the Nash bargaining solution satisfies

$$w^* f'(w^*) = p_0 \text{ and } p^* = \frac{p_0 + f(w^*)}{2w^*}.$$

(**c**) Find the Nash bargaining solution for $f(w) = aw + b, a > 0$.

THE SHAPLEY VALUE WITH MAPLE

The following Maple commands can be used to calculate the Shapley value of a cooperative game. All you need to do is to let S be the set of numbered players, and define the characteristic function as v. The list $M = [M[k]]$ consists of all the possible coalitions.

```
>restart:with(combinat):S:={1,2,3,4};
>L:=powerset(S):M:=convert(L,list):M:=sort(M,length);K:=nops(L);
># Define the characteristic function
>for k from 1 to K do if nops(M[k])<=1 then v(M[k]):=0; end if;end do;
v({1,2}):=0:v({1,3}):=0:v({2,3}):=0:v({1,4}):=5:v({2,4}):=10:v({3,4}):=0:
v({1,2,3}):=0:v({1,3,4}):=25:v({2,3,4}):=30:v({1,2,4}):=35:v({1,2,3,4}):=55:
># Calculate Shapley
> for i from 1 to nops(S) do
  x[i]:=0:
     for k from 1 to K do
        if member(i,M[k]) and nops(M[k])>=1 then
           x[i]:=x[i]+(v(M[k])-v(M[k] minus {i}))*
                ((nops(M[k])-1)!*(nops(S)-nops(M[k]))!)/nops(S)!
        end if;
     end do;
  end do:

> for i from 1 to nops(S) do lprint(shapley[i]=x[i]); end do;
```

BIBLIOGRAPHIC NOTES

The pioneers of the theory of cooperative games include L. Shapley, W. F. Lucas, M. Shubik, and many others, but may go back to Francis Edgeworth in the 1880s. It received a huge boost in the publication in 1944 of the seminal work by von Neumann and Morgenstern [26] and then again in a 1953 paper by L. Shapley in which he introduced the Shapley value of a cooperative game.

There are many very good discussions on cooperative game theory, and they are listed in the references. The conversion of any N-person non-zero sum game to characteristic form is due to von Neumann and Morgenstern, which we follow, as presented in references by Wang [28] and Jones [7]. Example 5.9 (used here with permission of Mesterton-Gibbons) is called the "log hauling problem" by Mesterton-Gibbons [15] as a realistic example of a game with empty core. It is a good candidate to illustrate how the least core with a positive ε^1 results in a fair allocation in which all the players are dissatisfied with the allocation. The use of Maple to plot and animate $C(\varepsilon)$ as ε varies is a great way to show what is happening with the level of dissatisfaction and the resulting allocations. For the concept of the nucleolus we follow the sequence in Wang's book [28], but this is fairly standard. The allocation of costs and savings games can be found in the early collection of survey papers in reference [13]. Problem 5.19 is a modification of a scheduling problem known as the "antique dealer's problem" in Mesterton-Gibbon's fine book [15], in which we may consider savings games in **time** units rather than monetary units.

The Shapley value is popular because it is relatively easy to compute but also because, for the most part, it is based on a commonly accepted set of economic principles. The United Nations Security Council example (Example 5.19) has been widely used as an illustration of quantifying the power of members of a group. The solution given here follows the computation by Jones [7]. Example 5.20 is adapted from an example due to Aliprantis and Chakrabarti [1] and gives an efficient way to compute the Shapley allocation of expenses to multiple users of a resource, and taking into account the ability to pay and requirement to meet the expenditures.

The theory of bargaining presented in Section 5.4 has two primary developers: Nash and Shapley. Our presentation for finding the optimal threat strategies in section 5.4.2 follows that in Jones' book [7]. The alternative method of bargaining using the KS solution is from Aliprantis and Chakrabarti [1], where more examples and much more discussion can be found. Our union versus management problem (Problem 5.31) is a modification of an example due to Aliprantis and Chakrabarti [1].

We have only scratched the surface of the theory of cooperative games. Refer to the previously mentioned references and the books by Gintis [4], Rasmussen [22], and especially the book by Osborne [19], for many more examples and further study of cooperative games.

Institute for Advanced Study at Princeton New Jersey (along with A. Einstein) and helped to make it the most prestigious research institute in the world.

The von Neumann minimax theorem was proved in 1928 and was a major milestone in the theory of games. Von Neumann continued to think about games and wrote the classic *Theory of Games and Economic Behavior* [26] (written with economist Oskar Morgenstern[1]) in 1944. It was a breakthrough in the development of economics using mathematics and in the mathematical theory of games. His contributions to pure mathematics fills volumes. The cleverness and ingenuity of his arguments amaze mathematicians to this day.

Von Neumann was one of the most creative mathematicians of the twentieth century. In a century in which there were many geniuses and many breakthroughs, von Neumann contributed more than his fair share. He ranks among the greatest mathematicians of all time for his depth, breadth, and scope of contributions. In addition, and perhaps more importantly, von Neumann was famous for the parties he hosted throughout his lifetime and in the many places he lived. He was an aristocratic *bon vivant* who managed several careers even among the political sharks of the cold war era without amassing enemies. He was well liked by all of his colleagues and lived a contributory life.

If you want to read a very nice biography of John von Neumann read MacRae's excellent book [14].

JOHN FORBES NASH. John Forbes Nash, Jr. was born June 13, 1928 in Bluefield, West Virginia. He was awarded the Nobel Prize in Economics (formally the 1994 Bank of Sweden Prize in Economic Sciences), which he shared with the mathematical economists and game theorists Reinhard Selten and John Harsanyi. This is the most prestigious prize in economics but certainly not the only prize won by Nash. In 1978 Nash was awarded the John Von Neumann Theory Prize for his invention of noncooperative equilibria, now called **Nash equilibria** and in 1999 Nash was awarded the Leroy P. Steele Prize by the American Mathematical Society.

Nash continues to work on game theory, and his contributions to the theory of games has certainly been as profound as that of von Neumann and others. Like von Neumann, Nash is truly a pure mathematician with a very creative, penetrating, and inquisitive mind, with fundamental contributions to differential geometry, global analysis, and partial differential equations. Despite the power and depth of his thinking and mathematical ability, between 1945 and 1996, he published only 23 papers (most of which contain major and fundamental results). The primary reason for the unproductive period in his life is the illness that he suffered, and suffers to

[1]Born January 24, 1902, in Germany and died July 26, 1977, in Princeton, NJ. Morgenstern's mother was the daughter of the German emperor Frederick III. He was a professor at the University of Vienna when he came to the United States on a Rockefellar Foundation fellowship. In 1938, while in the US, he was dismissed from his post in Vienna by the Nazis and became a professor of economics at Princeton University where he remained until his death.

The game is symmetric and has solution $X^* = (0, \frac{5}{11}, \frac{5}{11}, 0, \frac{1}{11}) = Y^*, v = 0$.

2.22 $A = \begin{bmatrix} 0 & 2 & -3 & 0 \\ -2 & 0 & 0 & 3 \\ 3 & 0 & 0 & -4 \\ 0 & -3 & 4 & 0 \end{bmatrix}$, $X^* = Y^* = (0, \frac{4}{7}, \frac{3}{7}, 0)$, for example.

2.23 The matrix B is given by

$$B = \begin{bmatrix} 0 & 0 & 5 & 2 & 6 & -1 \\ 0 & 0 & 1 & \frac{7}{2} & 2 & -1 \\ -5 & -1 & 0 & 0 & 0 & 1 \\ -2 & -\frac{7}{2} & 0 & 0 & 0 & 1 \\ -6 & -2 & 0 & 0 & 0 & 1 \\ 1 & 1 & -1 & -1 & -1 & 0 \end{bmatrix},$$

and will have $v(B) = 0$, and $P = Q = \left(\frac{5}{53}, \frac{6}{53}, \frac{3}{53}, \frac{8}{53}, 0, \frac{31}{53}\right)$. Then $b = (5 + 6)/53 = \frac{11}{53}$ and $b = (3 + 8 + 0)/53 = \frac{11}{53}$, so $x_1 = \frac{5}{11}, x_2 = \frac{6}{11}$ and $y_1 = \frac{3}{11}, y_2 = \frac{8}{11}, y_3 = 0$. Also, $v(A) = \gamma/b = \frac{31}{53}/\frac{11}{53} = \frac{31}{11}$.

2.24 **(a)**$X^* = (\frac{3}{5}, \frac{2}{5}), Y^* = (\frac{1}{5}, 0, 0, \frac{4}{5}, 0), v = \frac{8}{5}$.
(b) $X^* = (\frac{21}{53}, \frac{24}{53}, \frac{8}{53}, 0), Y^* = (\frac{23}{53}, \frac{4}{53}, \frac{26}{53}, 0), v = \frac{4}{53}$;
(c) $X^* = (\frac{9}{55}, 0, \frac{1}{5}, \frac{7}{11}), Y^* = (\frac{34}{55}, \frac{2}{11}, \frac{1}{5}, 0), v = \frac{51}{55}$.

2.25 $X^* = (\frac{47}{82}, \frac{10}{41}, \frac{15}{82}, 0), Y^* = (\frac{22}{41}, \frac{14}{41}, \frac{5}{41}, 0, 0), v = \frac{13}{41}$.

2.26 The pure strategies are labeled plane(P), highway(H), roads(R), for each player. The drug runner chooses one of those to try to get to New York, and the cops choose one of those to patrol. The game matrix in which the drug runner is the row player, becomes

$$A = \begin{bmatrix} -18 & 150 & 150 \\ 100 & 24 & 100 \\ 80 & 80 & 35 \end{bmatrix}.$$

For example, if drug runner plays H and cops patrol H, the drug runner's expected payoff is $(-90)(0.4) + (100)(0.6) = 24$. The saddle point is

$$X^* = (0.144, 0.3183, 0.5376) \text{ and } Y^* = (0.4628, 0.3651, 0.1721).$$

The drug runners should use the back roads more than half the time, but the cops should patrol the back roads only about 17% of the time.

2.27 The game matrix is

$$\begin{bmatrix} -0.40 & -0.44 & -0.6 \\ -0.28 & -0.40 & -0.2 \\ -0.2 & -0.6 & 0 \end{bmatrix}$$

3.21

$$\begin{array}{lll} X_1 = (1,0) & Y_1 = (1,0,0) & E_{\rm I} = 2, E_{\rm II} = 1 \\ X_2 = (1,0) & Y_2 = (\frac{1}{2},0,\frac{1}{2}) & E_{\rm I} = \frac{1}{2}, E_{\rm II} = 1 \\ X_3 = (0,1) & Y_3 = (0,0,1) & E_{\rm I} = 1, E_{\rm II} = 3 \end{array}$$

3.22 Take $B = -A$. The Nash equilibrium is $X^* = (\frac{5}{8}, \frac{3}{8}, 0), Y^* = (0, \frac{5}{8}, \frac{3}{8})$, and the value of the game is $v(A) = \frac{1}{8}$.

3.23 The objective function is $f(x,y,p,q) = 7x + 7y - 6xy - 6 - p - q$ with constraints $2y - 1 \leq p, 5y - 3 \leq p$, $2x - 1 \leq q, 5x - 3 \leq q$, and $0 \leq x, y \leq 1$.

$$\begin{array}{lll} X_1 = (1,0) & Y_1 = (0,1) & E_{\rm I} = -1, E_{\rm II} = 2 \\ X_2 = (0,1) & Y_2 = (1,0) & E_{\rm I} = 2, E_{\rm II} = -1 \\ X_3 = (\frac{2}{3},\frac{1}{3}) & Y_3 = (\frac{2}{3},\frac{1}{3}) & E_{\rm I} = \frac{1}{3}, E_{\rm II} = \frac{1}{3} \end{array}$$

3.24 $X_1 = \left(\frac{1}{2}, \frac{1}{3}, \frac{1}{6}\right), Y_1 = \left(\frac{6}{13}, \frac{5}{13}, \frac{2}{13}\right), E_{\rm I} = \frac{10}{13},\ E_{\rm II} = 1$. $X_2 = \left(\frac{3}{4}, 0, \frac{1}{4}\right) = Y_2$ with payoffs $E_{\rm I} = \frac{5}{4}, E_{\rm II} = \frac{3}{2}$. $X_3 = Y_3 = (0,1,0)$.

3.26 The matrices are

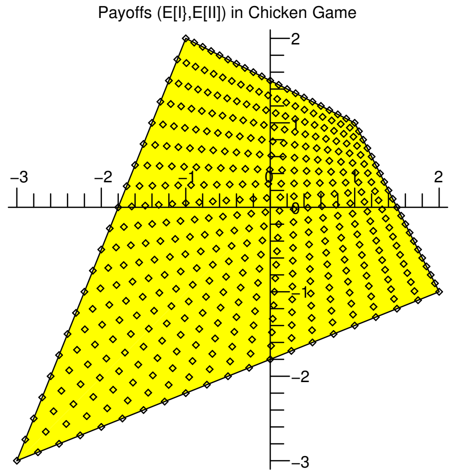
$$A = \begin{bmatrix} 1.20 & -0.56 & -0.88 & -1.2 \\ 1.24 & -0.40 & -1.44 & -1.6 \\ 0.92 & -0.04 & -1.20 & -1.8 \\ 0.6 & -0.2 & -0.6 & -2 \end{bmatrix}, B = \begin{bmatrix} 0.64 & 0.92 & 0.76 & 0.6 \\ -0.28 & 0.16 & -0.12 & -0.2 \\ -0.44 & 0.28 & 0.04 & -0.6 \\ -0.6 & 0.2 & 0.6 & 0 \end{bmatrix}.$$

One Nash equilibrium is $X = (0.71, 0, 0, 0.29), Y = (0, 0, 0.74, 0.26)$. So Pierre fires at 10 paces about 75% of the time and waits until 2 paces about 25% of the time. Bill, on the other hand, waits until 4 paces before he takes a shot but 1 out of 4 times waits until 2 paces.

3.27 (**a**) The Nash equilibria are

$$\begin{array}{lll} X_1 = (1,0) & Y_1 = (0,1) & E_{\rm I} = -1, E_{\rm II} = 2 \\ X_2 = (0,1) & Y_2 = (1,0) & E_{\rm I} = 2, E_{\rm II} = -1 \\ X_3 = \left(\frac{2}{3},\frac{1}{3}\right) & Y_3 = \left(\frac{2}{3},\frac{1}{3}\right) & E_{\rm I} = \frac{1}{3}, E_{\rm II} = \frac{1}{3} \end{array}$$

They are all Pareto-optimal because it is impossible for either player to improve their payoff without simultaneously decreasing the other player's payoff, as you can see from the figure:



None of the Nash equilibria are payoff-dominant. The mixed Nash (X_3, Y_3) risk dominates the other two.

(**b**) The Nash equilibria are

$$X_1 = \left(\frac{1}{4}, \frac{3}{4}\right), Y_1 = (1,0), E_1 = 3, E_2 = 0,$$
$$X_2 = (0,1), Y_2 = (0,1), E_1 = E_2 = 1,$$
$$X_3 = (1,0) = Y_3, E_1 = E_2 = 3.$$

(X_3, Y_3) is payoff-dominant and Pareto-optimal.

(**c**)

$$X_1 = \left(\frac{1}{2}, \frac{1}{3}, \frac{1}{6}\right), Y_1 = \left(\frac{6}{13}, \frac{5}{13}, \frac{2}{13}\right), E_{\mathrm{I}} = \frac{10}{13}, E_{\mathrm{II}} = 1.$$
$$X_2 = \left(\frac{3}{4}, 0, \frac{1}{4}\right) = Y_2, E_{\mathrm{I}} = \frac{5}{4}, E_{\mathrm{II}} = \frac{3}{2}.$$
$$X_3 = Y_3 = (0,1,0), E_{\mathrm{I}} = 2, E_{\mathrm{II}} = 3.$$

Clearly X_3, Y_3 is payoff-dominant and Pareto-optimal. Neither (X_1, Y_1) nor (X_2, Y_2) are Pareto-optimal relative to the other Nash equilibria, but they each risk dominate (X_3, Y_3).

SOLUTIONS FOR CHAPTER 4

4.1 $u_i(q_1, \ldots, q_i, \ldots, q_N) = q_i(\sum_{j=1}^N q_j - c_i)$. The optimal quantities that each firm should produce is

$$q_i = \frac{1}{N+1}\left(NC_i - \sum_{j=1, j\neq i}^{N} c_j\right).$$

If $C_i = c, \ \ i = 1, 2, \ldots, N$, then $q_i = c/(N+1) \to 0, N \to \infty$.

4.2 With $u_1(q_1, q_2) = q_1(100 - 2\sqrt{(q_1 + q_2)}) - q_1 - 10$, and $u_2(q_1, q_2) = q_2(100 - 2\sqrt{(q_1 + q_2)}) - 2q_2 - 5$, we get $q_1^* = 795.88, q_2^* = 756.48$, and profits $u_1^* = 16066.77$, and $u_2^* = 14519.42$.

4.3 Profit for firm 1 is 10, compared with 16 or 7.11 if $c_2 = 5$ or $c_2 = 1$, resp.

4.4 We have to solve the system

$$q_1 = \frac{1}{2}[(\Gamma - q_2^1 - c_1)r_1 + (\Gamma - q_2^2 - c_1)r_2 + (\Gamma - q_2^3 - c_1)r_3],$$
$$q_2^i = \frac{1}{2}[\Gamma - q_1 - c^i], i = 1, 2, 3,$$

which has solution

$$q_1 = \frac{1}{3}\left[\Gamma - 2c_1 + r_1(c^1 - c^3) + r_2(c^2 - c^3)\right]$$
$$q_2^1 = \frac{1}{3}\left[\Gamma + c_1 - \frac{c^3}{2} + \frac{1}{2}r_1(c^3 - c^1) + \frac{r_2}{2}(c^3 - c^2)\right] - \frac{c^1}{2}$$
$$q_2^2 = \frac{1}{3}\left[\Gamma + c_1 - \frac{c^3}{2} + \frac{1}{2}r_1(c^3 - c^1) + \frac{r_2}{2}(c^3 - c^2)\right] - \frac{c^2}{2}$$
$$q_2^3 = \frac{1}{3}\left[\Gamma + c_1 + \frac{1}{2}r_1(c^3 - c^1) + \frac{r_2}{2}(c^3 - c^2)\right] - \frac{2c^3}{3}$$

The quantities with the information given are $q_1 = \frac{263}{8}$ for firm 1, and $q_2^1 = \frac{529}{16}, q_2^2 = \frac{521}{16}$, and $q_2^3 = \frac{497}{16}$.

4.6 **(b)** $q_2 = \frac{1}{2}(18 - q_1)$; **(c)** $q_1 = 8$; **(d)** $q_2 = 5$.

4.7 $q_2(q_1) = (\Gamma - q_1 - 2c_2 + c_3)/3, q_3 = (\Gamma - q_1 - 2c_3 + c_2)/3$. Then $q_1 = (\Gamma + c_2 + c_3 - 3c_1)/2$, and $u_1(q_1, q_2, q_3) = (\Gamma + c_2 + c_3 - 3c_1)^2/12$.

4.9 $q_1^0 = 91.67, p = 8.32, u_1 = 569.8$.

(**b**) For the data given in the problem $s^* = 338.66, g^* = 357.85$.

4.23 The cumulative distribution function is $F(p) = -2p^3 + 3p^2, 0 < p < 1$. The interior solution of $1 - F(p) - pf(p) = 0$ is $p^* = 0.422$, so the reserve price should be set at 42.2% of the normalized range of prices. Notice that even though the density is symmetric around $p = \frac{1}{2}$, the optimal reserve price is not 0.5. The Maple commands to solve are

```
> restart: f:=x->6*x*(1-x);
> F:=x->int(f(y),y=0..x);
> fsolve(1-F(x)-x*f(x)=0,x);
```

4.25 The expected payoff of a bidder with valuation v who makes a bid of b is given by

$$u(b) = vProb(b \text{ is high bid}) - b = vF(\beta^{-1}(b))^{N-1} - b = v\beta^{-1}(b)^{N-1} - b.$$

Differentiate, set to zero, and solve to get $\beta(v) = ((N-1)/N)v^N$.

Since all bidders will actually pay their own bids and each bid is $\beta(v) = (N - 1/N)v^N$, the expected payment from each bidder is

$$E[\beta(V)] = \frac{N-1}{N}\int_0^1 v^N dv = \frac{N-1}{N(N+1)}.$$

Since there are N bidders, the total expected payment to the seller will be $(N-1)/(N+1)$.

SOLUTIONS FOR CHAPTER 5

5.2 (**a**) $v(1) = value(A) = \frac{8}{5}, v(2) = value(B^T) = \frac{8}{5}, v(12) = 6, v(\emptyset) = 0$.

(**b**) The core is $C(0) = \{(6 - x_2, x_2) \mid \frac{8}{5} \le x_2 \le \frac{22}{5}\}$.

(**c**) The least core is $C\left(-\frac{7}{5}\right) = \{(3,3)\}$.

5.3 In normalized form simply divide each number by 13 : $\vec{x}$ unnormalized=$\left\{\left(\frac{13}{4}, \frac{33}{8}, \frac{33}{8}, \frac{3}{2}\right)\right\}$.

5.4 $v(\emptyset) = 0, v(1) = \frac{3}{5}, v(2) = 2, v(3) = 1, v(12) = 5, v(13) = 4, v(23) = 3, v(123) = 16$.

5.5 $\varepsilon^1 = \frac{1}{3}$ and $C\left(\frac{1}{3}\right) = \left\{\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)\right\}$.

5.6 $\varepsilon^1 = -1$ least core is $C(-1) = \{\vec{x} = (2,2,0)\}$.

5.7 Suppose $\vec{x} \in C(0)$ so that $e(S, \vec{x}) \le 0, \forall S \subsetneq N$. Take the single player coalition $S = \{i\}$ so $v(i) + v(N - i) = v(N)$. Since the game is essential, $v(N) > \sum_{i=1}^{n} v(i)$.

Since $\vec{x}$ is in the core, we have

$$v(N) > \sum_{i=1}^{n} v(i) = \sum_{i=1}^{n} v(N) - v(N-i) = nv(N) - \sum_{i=1}^{n} v(N-i),$$
$$\Longrightarrow$$
$$v(N)(n-1) < \sum_{i=1}^{n} v(N-i) \le \sum_{i=1}^{n} \sum_{j \neq i} x_j = \sum_{i=1}^{n} v(N) - x_i$$
$$= nv(N) - \sum_{i=1}^{n} x_i = (n-1)v(N) \Rightarrow\Leftarrow .$$

5.8 Since the game is inessential, $v(N) = \sum_{i=1}^{n} v(i)$. It is obvious that $\vec{x} = (v(1), \ldots, v(n)) \in C(0)$. If there is another $\vec{y} \in C(0), \vec{y} \neq \vec{x}$, there must be one component $y_i < v(i)$ or $y_i > v(i)$. Since $\vec{y} \in C(0)$, the first possibility cannot hold and so $y_i > v(i)$. This is true at any j component of $\vec{y}$ not equal to $v(j)$. But then, adding them up gives $\sum_{i=1}^{n} y_i > \sum_{i=1}^{n} v(i) = v(N)$, which contradicts the fact that $\vec{y} \in C(0)$.

5.9 Suppose $i = 1$. Then

$$x_1 + \sum_{j \neq 1} x_j = v(N) = v(N-1) \le \sum_{j \neq 1} x_j,$$

and so $x_1 \le 0$. But since $-x_1 = v(1) - x_1 \le 0$, we have $x_1 = 0$.

5.11 Let $\vec{x} \in C(0)$. Since $v(N-1) \le x_2 + \cdots + x_n = v(N) - x_1$, we have $x_1 \le v(N) - v(N-1)$. In general, $x_i \le v(N) - v(N-i), 1 \le i \le n$. Now add these up to get $v(N) = \sum_i x_i \le \sum_i \delta_i < v(N)$, which says $C(0) = \emptyset$.

5.13 The core is

$$C(0) = \{(x_1, x_2, 16 - x_1 - x_2) : \frac{3}{5} \le x_1 \le 13, 2 \le x_2 \le 12, 5 \le x_1 + x_2 \le 15\}.$$

The least core: $\varepsilon^1 = -\frac{62}{15}, C(\varepsilon^1) = \left\{\left(\frac{71}{15}, \frac{92}{15}, \frac{77}{15}\right)\right\}$.

5.14 $q = \frac{1}{2}\left(\frac{2}{5} + \frac{3}{10} + \frac{3}{10}\right) = \frac{1}{2}$. The characteristic function is $v(i) = 0, v(12) = v(13) = v(23) = 1, v(123) = 1$.

5.15 (**b**) To see why the core is empty, show first that it must be true $x_1 + x_2 = -2$, and $x_3 + x_4 = -2$. Then, since $-1 \le x_1 + x_2 + x_3 = -2 + x_3$, we have $x_3 \ge 1$. Similarly $x_4 \ge 1$. But then $x_3 + x_4 \ge 2$ and that is a contradiction.

(**c**) A coalition that works is $S = \{12\}$.

5.16 $X^1 = C(-\frac{1}{10}) = \{x_1 + x_2 = \frac{9}{10}, \frac{4}{10} \leq x_1, \frac{2}{10} \leq x_2\}$. The next least core is $X^2 = C(-\frac{1}{4}) = \{(\frac{11}{20}, \frac{7}{20}, \frac{2}{20})\}$.

5.17 The least core is the set $C(-1) = \{x_1 = 1, x_2 + x_3 = 11, x_2 \geq 1, x_3 \geq 2\}$. The nucleolus is the single point $\{(1, \frac{11}{2}, \frac{11}{2})\}$

5.18 For the least core $\varepsilon^1 = -\frac{1}{2}$:

$$\text{Least core} = X^1 = C(-\frac{1}{2}) = \{x_1 + x_2 = \frac{3}{2}, x_3 + x_4 = \frac{3}{2}, x_i \geq \frac{1}{2}, i = 1, 2, 3, 4,$$
$$x_2 + x_3 \geq \frac{3}{2}, x_1 + x_4 \geq \frac{3}{2}, x_1 + x_3 \geq \frac{5}{4}, x_2 + x_4 \geq \frac{1}{2},$$
$$x_1 + x_2 + x_3 \geq \frac{3}{2}, x_1 + x_2 + x_4 \geq \frac{3}{2}, x_1 + x_3 + x_4 \geq \frac{3}{2},$$
$$x_2 + x_3 + x_4 \geq \frac{3}{2}, x_1 + x_2 + x_3 + x_4 = 3\}.$$

Next X^2 has $\varepsilon^2 = 1$. X^3 has $\varepsilon^3 = 3$, and nucleolus=$\{(\frac{3}{4}, \frac{3}{4}, \frac{3}{4}, \frac{3}{4})\}$.

5.19 (**a**) The characteristic function is the number of hours saved by a coalition. $v(i) = 0$, and

$$v(12) = 4, v(13) = 4, v(14) = 3, v(23) = 6, v(24) = 2, v(34) = 2,$$

$$v(123) = 10, v(124) = 7, v(134) = 7, v(234) = 8, v(1234) = 13.$$

(**b**) Nucleolus=$\{(\frac{13}{4}, \frac{33}{8}, \frac{33}{8}, \frac{3}{2})\}$ with units in hours. The least core is

$$X^1 = C(-\frac{3}{2}) = \{x_1 + x_2 + x_3 = \frac{23}{2}, x_4 = \frac{3}{2},$$
$$x_1 + x_2 + x_3 + x_4 = 13, x_1 + x_2 + x_4 \geq \frac{17}{2},$$
$$x_2 + x_3 + x_4 \geq \frac{19}{2}, x_1 \geq \frac{3}{2}, x_2 \geq \frac{3}{2},$$
$$x_1 + x_2 \geq \frac{11}{2}, x_3 \geq \frac{3}{2}, x_1 + x_3 \geq \frac{11}{2},$$
$$x_2 + x_3 \geq \frac{15}{2}, x_1 + x_4 \geq \frac{9}{2}, x_2 + x_4 \geq \frac{7}{2},$$
$$x_3 + x_4 \geq \frac{7}{2}, x_1 + x_3 + x_4 \geq \frac{17}{2}\}$$

The next least core, which will be the nucleolus, is $X^2 = \{(\frac{13}{4}, \frac{33}{8}, \frac{33}{8}, \frac{3}{2})\}$ with $\varepsilon^2 = 10$.

(**c**) The schedule is set up as follows: (i) Curly works from 9:00 to 11:52.5, (ii) Larry works from 11:52.5 to 1:45, (iii) Shemp works from 1:45 to 3:30, and (iv) Moe works from 3:30 to 5:00.

5.20 The characteristic function for the **savings game** is $v(\emptyset) = 0, v(i) = 0, v(1234) = 22 - 8.5$, and

$$\begin{aligned} v(12) &= 13 - 7.5, \quad v(13) = 11 - 7, \quad v(14) = 12 - 7.5, \\ v(23) &= 10 - 6.5, \quad v(24) = 11 - 6.5, \quad v(34) = 9 - 5.5, \\ v(123) &= 17 - 7.5, \quad v(124) = 18 - 8, \quad v(134) = 16 - 7.5, \\ v(234) &= 15 - 7. \end{aligned}$$

The least core is

$$\begin{aligned} X^1 = C(-1.125) = \{ & x_1 + x_2 + x_3 + x_4 = 13.5, x_1 + x_2 + x_3 \geq 10.625, \\ & x_1 + x_2 + x_4 \geq 11.125, x_1 + x_3 + x_4 \geq 9.625, \\ & x_2 + x_3 + x_4 \geq 9.125, x_1 \geq 1.125, x_2 \geq 1.125, x_1 + x_2 \geq 6.625, \\ & x_3 \geq 1.125, x_1 + x_3 \geq 5.125, x_2 + x_3 \geq 4.625, \\ & x_4 \geq 1.125, x_1 + x_4 \geq 5.625, x_2 + x_4 \geq 5.625, x_3 + x_4 \geq 4.625\} \end{aligned}$$

$X^2 = \{(4.375, 3.875, 2.375, 2.875)\}$, so this is the nucleolus.

5.21 The characteristic function is $v(i) = 0, v(12) = 100, v(13) = 130$, $v(23) = 0, v(123) = 130$.The Shapley value is $\{(\frac{245}{3}, \frac{50}{3}, \frac{95}{3})\}$, and this point is not in $C(0)$.

The nucleolus of this game is $\{(115, 0, 15)\}$.

5.22

	player A	player B	player C
ABC	25	65	10
ACB	25	65	10
BAC	50	40	10
BCA	50	40	10
CAB	35	65	0
CBA	50	50	0
Total	235	325	40

5.23 Shapley value=$(\frac{4}{3}, \frac{16}{3}, \frac{16}{3})$.

5.24 Shapley value=$\{\frac{10}{3}, \frac{23}{6}, \frac{23}{6}, 2\}$. The hours of work for each player using the Shapley value are as follows: (i) Curly 9:00 to 12:10, (ii) Shemp from 12:10 to 1:50, (iii) Larry from 1:50 to 4:00, and (iv) Moe from 4:00 to 5:00.

5.25 (**a**) $C(0) = \emptyset$.
(**b**) $X^1 = C(2) = \{(1, 1, 1, 1)\}$.
(**c**) Shapley value=$\{(1, 1, 1, 1)\}$.
(**d**) Shapley value for original game is $(-1, -1, -1, -1)$.

5.26 (**a**) $v(1) = 100, v(2) = v(3) = 0, v(12) = 150, v(13) = 160, v(23) = 0, v(123) = 160$.

In order for X_3 to be an ESS, we need

$$\frac{1}{2} > \frac{1}{2} + \frac{p}{2} - 2px + 2px^2,$$

which becomes $0 > 2p(x - \frac{1}{2})^2$, for $0 < p < p_x$. This is clearly impossible, so X_3 is not an ESS.

6.2 There are three Nash equilibria $X_1 = Y_1 = (1, 0)$, $X_2 = Y_2 = (0, 1)$, and the mixed $X_3 = Y_3 = (\frac{2}{3}, \frac{1}{3})$. The first two are ESSs. For X_3, $u(\frac{2}{3}, \frac{2}{3}) = \frac{2}{3}$, $u(x, \frac{2}{3}) = \frac{2}{3}$. Is $u\left(\frac{2}{3}, x\right) = \frac{2}{3} > u(x, x)$? No, because $\frac{2}{3} > x^2 + 2(1-x)^2$ is false for all $0 < x < 1$.

6.3 The only symmetric (nonstrict) Nash is $(X^* = (0, 1), X^*)$. Then $u(0, 0) = 1$, $u(x, 0) = 1$, $u(x, x) = -2x^2 + 5x + 1$, and $u(0, x) = 5x + 1$. Hence, $u(0, 0) = 1 = u(x, 0)$ and $u(x, x) < u(0, x)$, for any $0 < x \leq 1$. This means that $X^* = (0, 1)$ is an ESS.

6.4 The Nash equilibria and their payoffs are shown in the following table; they are all symmetric.

X^*	$u(X^*, X^*)$
$(1, 0, 0)$	2
$(0, 1, 0)$	2
$(0, 0, 1)$	2
$(\frac{3}{4}, \frac{1}{4}, 0)$	$\frac{5}{4}$
$(\frac{1}{4}, 0, \frac{3}{4})$	$\frac{5}{4}$
$(0, \frac{3}{4}, \frac{1}{4})$	$\frac{5}{4}$
$(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$	$\frac{2}{3}$

For $X^* = (1, 0, 0)$ you can see this is an ESS because it is strict. Consider next $X^* = (\frac{3}{4}, \frac{1}{4}, 0)$. Since $u(Y, X^*) = \frac{5}{4}(y_1 + y_2) - y_3/2$, the set of best response strategies is $Y = (y, 1-y, 0)$. Then $u(Y, Y) = 4y^2 - 4y + 2$,and $u(X^*, Y) = -\frac{1}{4} + 2y$. Since it is **not** true that $u(Y, Y) < u(X^*, Y)$, for all best responses $Y \neq X^*$, X^* is not an ESS.

6.5 (**a**) There is a unique Nash, strict and symmetric ESS$= (0, 1)$ if $a < 0, b > 0, = (1, 0)$ if $b < 0, a > 0$.

(**b**) Three Nash equilibria, all symmetric, NE$= (1, 0), (0, 1), X, X = (b/(a + b), a/(a + b))$. Both $(1, 0), (0, 1)$ are strict, so $(1, 0), (0, 1) \in ESS$. The mixed X is not an ESS since $E(1, 1) = a > ab/(a + b) = E(X, 1)$ so $ESS = \{(0, 1), (1, 0)\}$.

(**c**) Two strict asymmetric Nash Equilibria, one symmetric Nash Equilibrium $X = (c = b/(a+b), a/(a+b))$, but now X is ESS since

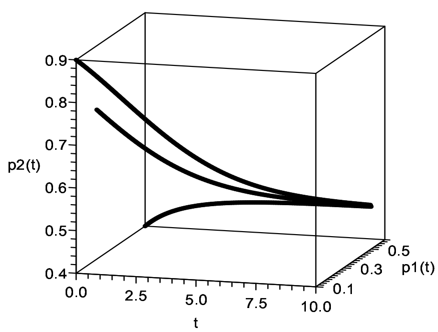
$$E(X,Y) = cay_1 + (1-c)by_2 = ab/(a+b)$$

and for every strategy $Y \neq X,\quad E(Y,Y) = ay_1^2 + by_2^2 < ab/(a+b) = E(X,Y)$, so X is the ESS.

6.7 The pure Nash equilibria are clearly equivalent. For the interior mixed Nash, the calculus method shows that the partial in the appropriate variables of the payoff functions lead to equations for the Nash equilibrium independent of a, b. You may also calculate directly that $E'(X,Y) = XA'Y^T = XAY^T - (a \quad b)Y^T = E(X,Y) - (a \quad b)Y^T$. Therefore, $E'(X^*,Y^*) \geq E'(X,Y^*)$ for all X, if and only if $E(X^*,Y^*) \geq E(X,Y^*)$, for all X.

6.8 (**b**) The three Nash equilibria are $X_1 = (\frac{1}{2}, \frac{1}{2}) = Y_1$, and the two nonsymmetric Nash points $((0,1),(1,0))$ and $((1,0),(0,1))$. So only X_1 is a possible ESS.

(**c**) From the following figure you can see that $(p_1(t), p_2(t)) \to (\frac{1}{2}, \frac{1}{2})$ as $t \to \infty$ and conclude that (X_1, X_1) is an ESS. Verify directly using the stability theorem that it is asymptotically stable.



The figure shows trajectories starting from three different initial points. In the three-dimensional figure you can see that the trajectories remain in the plane $p_1 + p_2 = 1$. The Maple commands used to find the stationary solutions, check their stability, and produce the graph are

```
> restart:with(DEtools):with(plots):with(LinearAlgebra):
> A:=Matrix([[3,2],[4,1]]); X:=<x1,x2>;
> Transpose(X).A.X;
> s:=expand(%);
> L:=A.X; f:=(x1,x2)->L[1]-s;g:=(x1,x2)->L[2]-s;
> solve({f(x1,x2)=0,g(x1,x2)=0},[x1,x2]);
```

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