EMILY PETERS' RESEARCH STATEMENT

1. Overview

My research concerns subfactors: inclusions of von Neumann algebras with trivial centers. My main project is the classification of small-index subfactors and the construction of exotic subfactors. Many examples of subfactors are known, as subfactors can be constructed from many already well-understood algebraic objects such as groups, quantum groups or conformal field theories. However, there are also exotic subfactors which do not arise in this way. Very few of these are known.

My results include:

- An alternate construction of Haagerup's exotic subfactor, using planar algebras [28].
- A planar-algebras construction of a new exotic subfactor, called the extended Haagerup subfactor [4].
- Contributing to the classification of subfactors with index less than 5 [25].
- Uniqueness of certain subfactors with index slightly more than 5 (work in progress) [23].

My future projects include:

- Proving theorems which make it possible to write an automated subfactor construction program in the 'FusionAtlas' (work in progress) [23, 22].
- Classifying subfactors up to an index higher than 5.
- Investigating the boundary between 'tame' and 'wild' behavior of subfactors.
- Proving diagrammatically that the higher Haagerup subfactors do not exist.

Throughout, "Questions" are motivational (and probably hard), while "Problems" are more approachable.

2. History

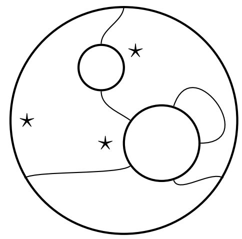
To understand von Neumann algebras, one can start with factors: von Neumann algebras having trivial center. Since von Neumann algebras have no two-sided ideals, every map between two von Neumann algebras is an inclusion; so understanding subfactors is equivalent to understanding all maps between factors.

In general, subfactors can be quite hard to get your hands on; it's good to have invariants to tell them apart. The first example of these was Jones' *index*, a real number measuring the relative size of the subfactor, which Jones [14] famously showed must lie in

$$\{4\cos^2(\frac{\pi}{n})|n = 3, 4, 5, \dots\} \cup [4, \infty].$$

The *planar algebra* of a subfactor, built from iterating Jones' basic construction, is a stronger invariant. It has a graded algebra structure, and it also has a diagrammatic structure, in which certain pictures can be interpreted as maps among the component algebras. This example led Jones to define more general planar algebras in [15]. Since then, planar algebras have been applied more broadly, not just to subfactors but to problems in knot theory [20, 21], statistical mechanics [17] and free probability [10].

Definition 2.1. *A* planar algebra *is a family of vector spaces* $\{V_i\}_{i\in\mathbb{Z}_{\geq 0}}$*, such that pictures like this*



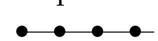
(i.e., a picture with an outside disk, some number of inside disks, marked points near the boundary of each disk, and non-crossing strings connecting the disks up) give a multilinear map

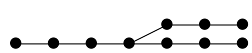
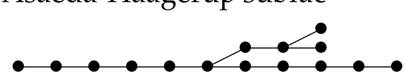
- *taking one input for each inner disk, coming from V_i (where i is the number of strands of that disk);*
- *giving an output in V_u (where u is the number of strands of the outer disk).*

For example, the above picture gives a map $V_2 \otimes V_5 \to V_3$.

Such pictures (considered up to isotopy) are called planar tangles. They have a natural composition, by putting one planar tangle into one of the inner disks of another, if the strings match up. The action of planar tangles on planar algebras must be compatible with this composition.

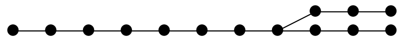
Although not all planar algebras come from subfactors, when they do, the process can be reversed; A result of Popa [29] constructs subfactors from planar algebras satisfying certain conditions. Thus many properties of subfactors are visible in planar algeras.

The *principal graph* of a subfactor is an invariant intermediate between the index and the planar algebra of a subfactor. It is a bipartite graph encoding the tensor product decomposition rules for the generator of the planar algebra of the subfactor. It recovers the index of the subfactor as the square of the Frobenius-Perron eigenvalue of the principal graph (or equivalently the square of the operator norm of its adjacency matrix). A subfactor with principal graph $A_\infty =$  is called trivial.

The question of which finite graphs can occur as principal graphs is very interesting. If a subfactor has index less than or equal to 4, its principal graph is a Dynkin or extended Dynkin diagram. Haagerup [11] showed in 1993 that there are only two infinite families of graphs and one stand-alone graph which could be principal graphs of (non-trivial) subfactors with index in the range $(4, 3+\sqrt{3})$. In 1999, Haagerup and Asaeda [1] gave constructions of two subfactors whose principal graphs are on this list. They are called the Haagerup and Asaeda-Haagerup subfactors, and have principal graphs  and  and indices $\frac{5+\sqrt{13}}{2} \approx 4.30278$ and $\frac{5+\sqrt{17}}{2} \approx 4.56155$, respectively. The Haagerup subfactor is the smallest example of an exotic subfactor

All but one of the graphs on Haagerup's list [11] can be proven not to exist. Bisch [5] demonstrated that one of the families of graphs would give non-associative fusion rules. Asaeda and Yasuda [2, 3] used number-theoretic results of Etingof, Nikshych and Ostrik [9] to show that the graphs in the Haagerup family beyond the first two cannot be principal graphs of subfactors either.

The classification of subfactors with index less than $3+\sqrt{3}$ was completed in 2009, when we constructed the "extended Haagerup" planar algebra in [4]. The extended Haagerup planar algebra has index ≈ 4.37720 and principal graph

, and currently has the highest supertransitivity (distance before a branch point) of any known subfactor.

3. Potential principal graphs and the FusionAtlas

We begin with a hard but motivational question.

Question 3.1. *Which bipartite graphs are principal graphs for subfactors?*

No simple answer to this question is known, but progress has been made in the case of small-index principal graphs. As described above, the classification of subfactors up to index $3 + \sqrt{3}$ was completed when the extended Haagerup subfactor was constructed. Recently, the classification has been extended to subfactors with index less than 5. Both of these classifications are stated in terms of principal graphs.

Theorem 3.2 (Han, Izumi, Jones, Morrison, Penneys, Snyder and Tener [12, 24, 25, 13, 27])**.** *The only subfactors with index in $[3 + \sqrt{3}, 5)$ are trivial subfactors, the unique subfactor subfactor with the GHJ 331 principal graph and the unique subfactor with the Izumi-Xu 2221 principal graph.*

The uniqueness of the Izumi-Xu 2221 subfactor was established by Han in [12]. The rest of this theorem was proved in a series of four papers.

In the first paper, Morrison and Snyder [24] used the FusionAtlas computer programs to narrow down the principal graphs which could occur for a subfactor with index between $3 + \sqrt{3}$ and 5. FusionAtlas [22] is a Mathematica package I have contributed to, designed to deal with principal graphs and planar algebras. One of FusionAtlas's main features is the enumeration of principal graphs.

Morrison and Snyder's list of potential principal graphs with index less than 5 contained 54 'vines' and five 'weeds.' Vines are graphs whose translates may be principal graphs (that is, everything in the vine family looks alike except for the initial segment, which may vary in length). Weeds are graphs whose translates and extensions may be principal graphs (that is, weeds can have further vertices and branches growing out of the right side of the graph too). A single weed therefore represents an infinite number of different vines; unsuprisingly, less information can be read off of a weed's graph.

In the second paper [25], I showed (with Morrison, Penneys, and Snyder) that three of the five weeds appearing in the list of [24] cannot be principal graphs of subfactors. The techniques we used to rule these weeds out include the "quadratic tangles test," (which uses details of the generators-and-relations presentation of the planar algebra associated a weed), and tests which rule out weeds based on the non-existence of connections on their principal graphs.

The remaining two weeds are ruled out in [13], which also established uniqueness of the GHJ 3311 subfactor. A uniform treatment which rules out all the graphs coming from the 54 vines, other than those already know to exist, is given in [27].

The classification up to index 5 is further progress towards identifying the boundary between 'tame' and 'wild' index values for subfactors.

Problem 3.3. *We say an index is* tame *if subfactors at that index can be classified by their principal graphs or planar algebras. An index is* wild *if such a classification is impossible. Is there a boundary between 'tame' and 'wild' indices? If so, where is it?*

The recently completed classification of subfactors up to index 5 shows that 5 is a tame index. On the other hand, wild behavior is seen at index 6; Bisch, Nicoara and

Popa show in [7] that there are infinitely many non-isomorphic index-6 subfactors with the same planar algebra.

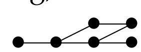
Conjecture 3.4. *The first wild index for subfactors is* $3+\sqrt{5} = 2\cdot\left(\frac{1+\sqrt{5}}{2}\right)^2 \approx 5.23607$.

One reason for hoping to see wild behavior at $3+\sqrt{5}$ is that this is the smallest index greater than four of a Fuss-Catalan subfactor [6]. Because the Fuss-Catalan planar algebras arise as a free product of Temperley-Lieb planar algebras, one can hope that various quotient planar algebras might yield infinitely many (possibly infinite-depth) planar algebras having this index.

We hope to make more progress on Problem 3.3 with further computer assistance from the FusionAtlas [22] in performing an exhaustive search for principal graphs below a certain index (ideally $3+\sqrt{5}$, although other values between 5 and $3+\sqrt{5}$ would also constitute progress).

Recently, Scott Morrison and I have implemented calculations of connections in graph planar algebras in the FusionAtlas. This has increased the number of tests available, which for instance makes it possible to classify one-supertransitive subfactors with index in the range $(5, 3+\sqrt{5})$. More importantly, this brings us a step closer to the following:

Work In Progress 3.5 (Morrison and Peters)**.** *Prove theorems making it possible to write a computer program which determines whether a finite graph P is the principal graph of a subfactor, by attempting to construct the associated planar algebra.*

This program will work by using a combined approach to construction (suggested by Jones), which finds both a generator of the planar algebra and a biunitary connection, and checks flatness of the generator (under the connection). This procedure is algorithmic, and involves solving a finite system of equations. However, it is important to note that as graph complexity (measured by index or supertransitivity) gets large, such a program quickly runs into time and memory constraints. Nevertheless, we believe that this approach is promising; recently, we showed [23] that there is a unique subfactor with principal graph  by finding a generator and a unique connection under which the generator is flat.

We are in the process of proving that a version of this construction works with bi-invertible, instead of biunitary, connections. Such a theorem will allow us to perform these computations over a number field, instead of using exact arithmetic, which makes many previously impossible calculations possible. This is what is meant above by "Prove theorems making it possible to write a computer program . . . ."

Previous methods of constructing subfactors have not been amenable to automating construction. If one tries to construct a planar algebra without the associated connection, one needs to know the generators-and-relations presentation of the planar algebra (and there is not an algorithm for reading this from a general principal graph). If one tries to construct a subfactor via connections only, one must check the "flatness" of the connection for all vertices in the principal graph, and this becomes computationally unfeasible much quicker than the combined construction approach.

4. Constructing subfactors with planar algebras

The FusionAtlas' graph enumerator (described in §3) can help us search for new exotic subfactors. Though we are not yet close to a complete classification up to index $3+\sqrt{5}$ (and such a classification is provably impossible past 6), a partial enumeration of graphs up to that index limit can be a good place to start looking for new examples of exotic subfactors.

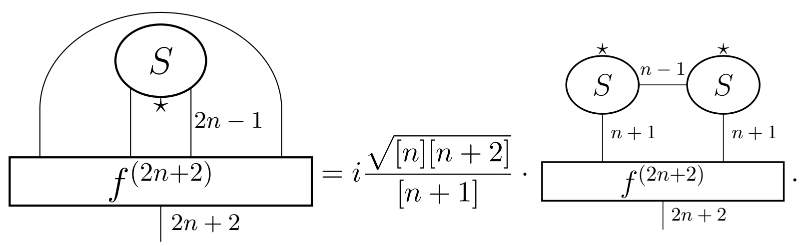
Planar algebras are proving to be a powerful tool with which to construct subfactors. Philosophically, the advantages of planar algebras are their ability to simultaneously encode different algebraic structures, and highlight symmetries. Practically speaking, Popa [29] showed how to construct, from a planar algebra (specifically, the equivalent structure called λ-lattices) with a given principal graph, a subfactor with that same principal graph. By a result of Jones-Penneys and Morrison-Walker [19, 26], one can always find the planar algebra of a finite-depth subfactor inside the graph planar algebra of its principal graph. In [16, 18], Jones developed tools for locating this sub-planar algebra.

In [28], I used these tools to prove the existence of the Haagerup planar algebra. These methods were extended in [4] to construct the planar algebra of the extended Haagerup subfactor, answering affirmatively the question of its existence.

Theorem 4.1 (Bigelow-Morrison-Peters-Snyder, [4])**.** *The extended Haagerup planar algebra exists.*

Theorem 4.2 (Bigelow-Morrison-Peters-Snyder, [4])**.** *Any planar algebra with principal graph in the Haagerup/extended Haagerup family (ie, in the Haagerup vine) must have a generator-and-relations presentation of a certain specified form.*

The presentation of Theorem 4.2 involves one generator, some very simple relations on that generator, and three relations similar to



(This relation says one can "pull a generator S past a string" by allowing it to double.)

Theorem 4.1 is proved by identifying a generator in the graph planar algebra and calculating its second through fourth moments (this is a computer-aided calculation of moments of large matrices). In combination with some linear algebra and skein theory involving Jones-Wenzl idempotents ($f^{(2n+2)}$ in the above pictures), these moments prove that this generator satisfies the relations given in the presentation of 4.2.

The presentation of Theorem 4.2 also raises hopes that one can do the following:

Problem 4.3. *Find a direct, diagrammatic proof that further extended Haagerup subfactors do not exist.*

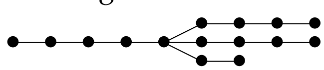
The further extended Haagerup subfactors are already known not to exist, but the proof is somewhat roundabout and relies on number theory. Because the associated planar algebras cannot exist either, there must be some diagram which can be evaluated in two different ways, to give two different answers. This would be a direct proof that the only planar algebra consistent with the relations dictated by the further extended Haagerup principal graph is trivial.

Another direction for exploration raised by Theorem 4.1 is the question of supertransitivity. Recall the supertranstitivity of a graph is the distance from the initial vertex to the first branch point. The extended Haagerup subfactor has supertransitivity 7, higher than any previously known supertransitivity of a (non-trivial) subfactor with index more than 4.

Recall from §3 that, unlike the classification up to $3+\sqrt{3}$, the classification up to 5 did not produce new examples of exotic subfactors. One might suspect that as one gets further from the classical region of index below 4, exotic subfactors would be easier to come by, but the evidence below index 5 seems to contradict this. It would be very interesting to know whether there is some sort of rigidity result obstructing exotic subfactors at higher indices. More specifically, one could ask

Question 4.4. *Do there exist (non-trivial) subfactors with index more than 4 of all supertransitivities?*

Many people believe that there should be a bound on supertransitivity. There is not much evidence for this, other than our inability to find subfactors of high supertransitivities. However, there is one result which does seem to point towards a bound on supertransitivity. Calegari, Morrison and Snyder [8] have shown that any vine (recall that a vine represents a family of graphs which vary only in their supertransitivity) can contain only finitely many prinicipal graphs of subfactors.

With help from FusionAtlas, I hope to produce further graphs to test this hypothesis on. Though the FusionAtlas graph enumerator can't produce a complete list of candidates when the upper limit on the index is more than 6, one can generate incomplete lists of candidate graphs, and look through them for graphs that "seem likely" to exist. Further testing, including attempts to construct the associated subfactors (either by hand or by computer), could generate further evidence for or against the existence of a bound on supertransitivity. The "4442" graph  is an example of a graph which has been found in this way, which is a good candidate to attempt to construct.

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