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Quantum dimension.

Outline:

0. Example.
1. Semisimple ribbon categories.

$1'$. Examples

2. Definition of quantum dimension.
3. Properties.
4. Computations for IPER of LSU(2).

0. Basic example. $V = \mathbf{C}^2$. Choose a basis $\{e_1, e_2\}$ and a dual basis $\{\epsilon_1, \epsilon_2\}$ for V^*. We have evaluation and coevaluation maps: $e: V \otimes V^* \to \mathbf{C}$, $i: \mathbf{C} \to V \otimes V^*$. $ci = 2$.

1. Let C be a complex-linear abelian category, which we will "give" some structure: $\otimes: C \times C \to C$, a bilinear functor, $\alpha_{U,V,W}: (U \otimes V) \otimes W \to U \otimes (V \otimes W)$ functorial isomorphism. If $I \in C$ is a monoidal unit, we also have monoidal isomorphisms $\lambda_V: I \otimes V \to V$ and $\rho_V: V \otimes I \to V$ for all $V \in C$. We also have $\mathrm{End}(V) = \mathbf{C}$.

Examples: Complex vector spaces, representations of $\mathrm{SU}(n)$. Tensor product gives a monoidal structure. Hilbert spaces also form an example.

Braiding: Functorial isomorphism $V \otimes W \to W \otimes V$.

Rigidity: $e_V: V^* \otimes V \to I$ and $i_V: I \to V \otimes V^*$ with the obvious properties. Similar conditions for the left dual.

Examples: Complex vector spaces.

Definition: A ribbon category is a rigid braided tensor category with functorial isomorphism $\delta_V: V \to V^{**}$ (can use left or right duals) such that $\delta_{V \otimes W} = \delta_V \otimes \delta_W$, $\delta_I = 1_I$, $\delta_{V^*} = (\delta_V^*)^{-1}$.

3. Quantum dimension. Let C be a semisimple ribbon category.

Definition: Let $V \in C$ and $f \in \mathrm{End}(V)$. Define $\mathrm{tr}(f)$ to be $I \to V \otimes V^* \to V \otimes V^* \to V^{**} \otimes V^* \to I$.

Property: $\mathrm{tr}(f \otimes g) = \mathrm{tr}(f)\,\mathrm{tr}(g)$.

Definition: The quantum dimension of $V \in C$ is defined to be $\mathrm{tr}(1_V)$.

4. Properties. Assume that there only finitely many simple objects. Dimension is real and positive.

5. Calculations. Denote $N_{f,g}^h = \dim \mathrm{Hom}(V_h, V_f \otimes V_g)$.

Theorem (Wassermann): $H_f \otimes H_g = \oplus N_{f,g}^h \,\mathrm{sign}(\sigma_n) H_{h'}$.

Corollary: $H \otimes H_f = \oplus_{g=f+1} H_g$.

Consider all IPER of $L\mathrm{SU}(2)$ of level l: $H_0, \ldots, H_l$. Define $d_i = \dim H_i$. Proposition: For all $0 \le i \le n$ we have $d_i = d_{l-i}$, $d_0 = d_l$, $d_1 d_i = d_{i-1} + d_{i+1}$. $d_1 d_{l-i} = d_{l-i-1} + d_{l-i+1}$. $d_{i-1} = d_{l-i+1}$, hence $d_{i+1} = d_{l-i-1}$.