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Modularity of the category of representations of a conformal net.

1. Introduction.
2. Two interval inclusions.
3. Modularity.

Goal: Let A be a completely rational conformal net.

1. Semi-simplicity: Every separable rep is completely reducible.
2. The number of unitary equivalence classes of irreducible representations is finite.
3. Every separable irreducible representation has finite statistical dimension.
4. The category of representations has a monoidal structure with simple unit and duals (conjugates) and a maximally non-degenerate braiding

A is completely rational conformal net. $A(I) \subset B(H_0)$. H_0 is the vacuum Hilbert space, $\Omega \in H_0$ is vacuum. U has a bounded positive energy representation of $\mathrm{PSU}(1,1)$.

(a) Show additivity.
(b) Split property: $I \subset\subset J$ implies $A(I) \to A(J)$ factors through type I factor.
(c) $\mu_2(A) < \infty$, $\mu_2 = [\hat{A}(E) : A(E)]$, $\hat{A}(E) = A(E')'$, where $E = I \cup J$, $\bar{I} \cap \bar{J} = \emptyset$.

H is separable, for any $I \subset \mathbf{I}$ there is $\rho \sim \pi$ that is localized in I $(\rho_{I'} = 1_{A(I')})$.

Conjugates: $\pi \sim \rho$ localized in I. Suppose P and Q are two other intervals. $\bar{\rho}_I(x) = J_P \rho_{r_Q I}(J_Q \times J_Q) J_P$ and $J_P A(P) J_P = A(P)'$.

Bisognano-Wichmann: $J_P \times J_P = U(r_P) \times U(r_P)^*$. $[\bar{\rho}_I]$ does not depend on P and Q.

Theorem: If π is separable and irreducible, then there exists $\bar{\pi}$. If π is Möbius, then $\bar{\pi}$ is Möbius covariant. $\rho \in [\pi]$ and $\bar{\rho} \in [\bar{\pi}]$.

2. Two interval inclusions.

Fact: If $N \subset M$ is irreducible ($N' \cap M = \mathbf{C}1$) and $[M : N] < \infty$, then there is a canonical endomorphism $\gamma(x) = J_N J_M x J_M J_N$. There is $\psi \in M$ such that $\psi x = \sigma(x)\psi$ for $x \in N$ if and only if there is $\sigma \in \mathrm{End}(N)$ such that $\sigma < \gamma|_N$ there is $U \in N$ such that $U(x) = \gamma(x)U$.

$E = I_1 \cup I_2$. $A(E) \subset \hat{A}(E) = A(E')'$. $\gamma_E\colon \hat{A}(E) \to A(E)$. Pick π_i an irreducible separable representation. $\rho_i \in [\pi_i]$. $\bar{\rho}_i \in [\bar{\pi}]$ and localized at I_2.

$R_i \in \mathrm{Hom}(1, \sigma = \rho_i\bar{\rho}_i)$, $\rho_i(\bar{\rho}_i(x))$. $R_i x = \rho_i(\bar{\rho}_i(x)) R_i$, $x \in A(E)$. $\oplus_{i \in \Gamma_E} \rho_i \bar{\rho}_i < \lambda_E = \gamma_E|_{A(E)}$.

$\sum_{i \in \Gamma_f} d(\rho_i)^2 \leq [\hat{A}(E), A(E)] = \mu_2$.

Fact: $\gamma(x) = \sum_i U_i \sigma_i(x) U_i^*$, σ_i irreducible. U_i such that $\sum_i U_i^* U_i = 1$, $U_j U_i^* = [i = j]1$. Every $x \in M$ is of the form $x = \sum x_i \psi_i$, $x_i \in N$.

3. Modularity.

Proposition. Every irreducible separable representation of a conformal net A has finite statistical dimension.

Proof sketch: Choose $\rho \in [\pi]$ localized at some interval, choose ρ' localized at some other disjoint interval. There is $u \in \mathrm{Hom}(\rho, \rho') \subset \hat{A}(E)$. $u = \sum u_i R_i$ implies $0 \neq u_i \in \mathrm{Hom}(\rho_i \rho, 1)$. $\rho \to \bar{\rho}_i \rho_i \rho \to \bar{\rho}_i$. $x \in \hat{A}(E)$, $x = \sum_{i \in \Gamma} x_i R_i$, $x_i \in A(E)$.

$\hat{A}(E) = A(E)$.

Braiding ρ, η localized irreps. $E(\rho, \eta) \in \mathrm{Hom}(\rho\eta, \eta\rho)$. $\eta_i \rho = \rho \eta_i$.

$T_{L/R} \in \mathrm{Hom}(\eta, \eta_{L,R})$. $E(\rho, \eta) = T_L^* \rho(T_L)$. $E(\eta, \rho)^* = T_R^* \rho(T_R)$.

Definition: ρ and η have trivial monodromy if $E(\rho, \eta) = E(\eta, \rho)^*$, which is equivalent to $E_M(\rho, \eta) = E(\rho, \eta) E(\eta, \rho) = 1$.

Definition: π is called finite if

1. π is a finite direct sum of irreps;
2. π has finite statistical dimension;
3. $\pi(C^*(A))'$ is finite.

Consider the category of all finite representations.

Definition: ρ is called degenerate if $E_M(\rho, \eta) = 1$ for all finite representations η $(\eta \in \Gamma)$.

Theorem: The center of the category of representations is trivial, thus it is modular.