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Clifford algebras and Fock representation.

$H$ is a complex Hilbert space. $\mathrm{Cliff}(H)$ is a complex unital *-algebra generated by $c\colon f \in H \mapsto c(f) \in \mathrm{Cliff}(H)$ such that $c(f)c(g) + c(g)c(f) = 0$ and $c(f)c(g)^* + c(g)^*c(f) = \langle f, g\rangle$.

Realize: $\mathrm{Cliff}(H) = T(H \oplus \bar{H})/(f \otimes f - \langle f, f\rangle$.

$\mathrm{Cliff}(H)$ has a representation on $\Lambda H$. More precisely, $\pi(c(f))(x) = f \wedge x$. The vector $\Omega = 1 \in \Lambda^0 H = \mathbf{C}$ is cyclic. We also have annihilation operators $a(f) = c(f)^*$, which can be explicitly expressed using the usual formula.

Proposition: $\Lambda H$ is irreducible as a representation of $\mathrm{Cliff}(H)$. Proof: An operator that commutes with creators and annihilators must be a multiple of identity.

Unitary structure: $V$ is a complex (not real!) Hilbert space. Denote the underlying real inner product by $(,)$. A unitary structure on $V$ is an element $J \in O(V)$ such that $J^2 = -1$.

$V_J$ is a complex Hilbert space, the underlying real Hilbert space being $V$ and the Hermitian product being $\langle v, w\rangle = (v, w) + i(v, Jw)$. Define a projection operator $P_J = (I - iJ)/2 \in \mathrm{End}(V_J)$.

We have the Fermionic Fock space $F_P = \Lambda(PH) \otimes \Lambda((1-P)H)^*$, where $H = V_J$. This is an irreducible representation of $\mathrm{Cliff}(V_J)$, more precisely $\pi_P(c(F)) = c(Pf) \otimes 1 + 1 \otimes c(((1-P)f)^*)^*$.

Theorem: (I. Segal and Shale) The Fock representations $\pi_P$ and $\pi_Q$ are unitarily equivalent if $P - Q$ is Hilbert-Schmidt.

An element $u \in U(H)$ gives an automorphism of $\mathrm{Cliff}(H)$ via $c(f) \mapsto c(uf)$. We say that this automorphism is implemented in $\pi_P$ if $\pi_P(c(uf)) = U\pi_P(c(f))U^*$ for some (almost unique) $U \in U(F_P)$.

Proposition: $u \in U(H)$ is implemented iff $[u, P]$ is Hilbert-Schmidt. All such $u$ constitute the restricted unitary group. We have a representation of this group on the projective unitary group of $F_P$.

If $u \in U(H)$ and $[u, P] = 0$, then $u$ is implemented in $F_P$ and is canonically quantized.

Representations of $LU(n)$. We have a Hilbert space $H = L^2(S^1) \otimes \mathbf{C}^n$. We have the Hardy subspace $H_{\geq 0}$ (nonnegative Fourier coefficients).

For $f \in C^\infty(S^1, \mathrm{End}(\mathbf{C}^n))$ define a multiplication operator $m(f)$ on $H$. Fact: $\|[P, m(f)]\|_2 \leq \|f'\|_2$.

In particular, $f \in LU(n)$ is implemented in $F_P$, hence we get a projective representation of $LU(n)$ on $F_P$ (the fundamental representation).

Consider $\mathrm{SU}_\pm(1,1)$, which is a double cover of Möbius group. We have

$$\mathrm{SU}_\pm(1,1) = \left\{ \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix} \mid |\alpha|^2 - |\beta|^2 = \pm 1 \right\}.$$

We have an action on the circle: $g(z) = (\alpha z + \beta)/(\bar{\beta} z + \alpha)$. An element $g$ of this group acts on $H = L^2(S_1) \otimes \mathbf{C}^n$ by $(V_g f)(z) = f(g^{-1}(z))/(\alpha - \bar{\beta} z)$. Note if $|z| < 1$ and $|\alpha| > |\bar{\beta}|$ then $(\alpha - \bar{\beta} z)^{-1}$ is holomorphic. Hence $V_g$ commutes with $P$ and the action is canonically quantized. If $F = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$, then $(V_F f)(z) = f(z^{-1})z^{-1}$, so $V_F P V_F = I - P$.

We also have a $U(1)$ action on $H$ given by multiplication by some fixed unitary complex number $z$. This action is canonically quantized, let $u_z$ be the action on $F_P$.

Proposition: If $\pi$ is the representation of $LSU(n)$ on $F_P$ and An element $g$ of this group acts on $H = L^2(S_1) \otimes \mathbf{C}^n$ $u_z$ is the $U(1)$-action on $F_P$, then $\pi(g) u_z \pi(g)^* = u_z$.