Classification of irreducible positive energy representations of loop groups.

Recall: $G = \mathrm{SU}(3)$. We accepted that we would classify reps of $\mathrm{SU}(3)$ in terms of highest weights of the adjoint representation. We thought of the highest weight as a signature in $\mathbf{Z}^3$: (f_1, f_2, f_3). Fix the adjoint representation $(1, 0, -1)$.

Theorem: The possible highest weights are $(f_1, \ldots, f_n)$ such that $f_1 \geq \cdots \geq f_n \geq 0$.

By adding $(a, \ldots, a)$ we set $f_n = 0$.

Fix $L\mathrm{SU}(3)$. What is the equivalent of a signature? What are the possible values?

Setup: (π, H) is an irreducible positive energy representation of LG at level l. Here π is a proective representation of the semidirect product of LG and the circle (honest representation on the circle) satisfying $H = \oplus_{n \geq 0} H(n)$.

Theorem: (1) $H(0)$ is an irreducible $\mathrm{SU}(n)$-modules; (2) The signature f of $H(0)$ satsifies $f_1 - f_n \leq l$; (3) If f is such a signature, there is an irreducible positive energy representation of LG with $H(0) = V_F$; (4) This representation is unique up to isomorphism.

Question: Why is $H(0)$ invariant under $\mathrm{SU}(n)$?

Proof: Recally that the polynomial Lie loop algebra consists of trigonometric polynomials in g. There exists a representation ρ of the semidiret product of the Lie loop algebra and $\mathbf{R}$ such that $\pi(\exp(x)) = \exp(\rho(x))$.

In particular, $H(i)$ is an $\mathrm{SU}(n)$-module.

Weight diagram for $L\mathrm{SU}(2)$ at level l is a paraboloid with points inside.