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Positive energy representations.

G is a compact connected Lie group, $\mathbf{T}$ is the circle as a group.

Definition. A PER of LG is a TVS E with (1) A projective representation of LG: $LG \to PU(E)$; (2) An intertwining action of $\mathbf{T}$: $R\colon \mathbf{T} \to \mathrm{GL}(E)$ such that $R_\theta U_\gamma R_\theta^{-1} = U_{R_\theta \gamma}$. (3) (Positive energy condition): Under the weight decomposition of E by $\mathbf{T}$ we have $E = \oplus_n E_n$, where $E_n = 0$ for $n < 0$ and E_n is finite-dimensional for all n.

Concrete examples: $G = \mathrm{SU}(n)$, $V = \mathbf{C}^n$. $H = L^2(S^1, V)$, $P\colon H \to H$ is the projection onto nonnegative Fourier modes. $F_P = \Lambda(PH) \otimes \Lambda(P^\perp H)^*$ (level 1 representation). $F_P^{\otimes l}$ is a level l representation.

Theorem. (Pressley-Segal, 9.3.1) A PER of LG is completely reducible into irreducible PERs, unitary, extend to holomorphic representations of $LG_{\mathbf{C}}$, and admit a projective intertwining action of $\mathrm{Diff}_+(S^1)$.

Motivation: States are elements of projective space, positive energy condition comes from physics, boundary conditions of holomorphic representations.

Definition: A projective unitary representation of a group G on a Hilbert space V is a continuous group homomoprhism of $G \to PU(V) = U(V)/\mathbf{T}$ with strong operator topology.

A projective representation gives a central extension: $1 \to \mathbf{T} \to \tilde{G} \to G \to 1$.

Idea 1: A PER leads to a central extension of LG: $1 \to \mathbf{T} \to \widetilde{LG} \to LG \to 1$. We have the first Chern class $c_1(\widetilde{LG} \to LG) \in H^2(LG, \mathbf{Z})$.

Idea 2: Extension of Lie algebras $0 \to \mathbf{R} \to \widetilde{Lg} \to Lg \to 0$. $(X, a) \in Lg \oplus \mathbf{R} = \widetilde{Lg}$; $[(X, a), (Y, b)] = ([X, Y], \omega(X, Y))$. Here ω is a skew-symmetric 2-cocycle satisfying Jacobi identity.

Proposition: (Pressley-Segal, 4.2.4) For g semisimple, every continuous G-invariant 2-cocycle ω for Lg has the form $\omega(X, Y) = (2\pi)^{-1} \int_{\mathbf{T}} \langle X(\theta), Y'(\theta) \rangle d\theta$, where $\langle\rangle$ is a g-invariant symmetric bilinear form on g.

Theorem (Pressley-Segal, 4.4.1) If G is simply connected, then

1. If ω is a 2-cocycle on Lg, then it defines a group extension if and only if $\omega/2\pi \in H^2(LG, \mathbf{Z})$;
2. If it does, then the extension is unique.
3. Such an ω is an integral multiple of ω_{basic}. (Here G is simple.)

Definition: The level of a central extension $\widetilde{LG}$ of LG is the integer l such that $\omega_{\widetilde{LG}} = l\omega_{\mathrm{basic}}$.

Definition: A PER of LG is an honest positive-energy representation of the semidirect product of $\widetilde{LG}$ and $\mathbf{T}$.

We have $L^{\mathrm{poly}} g \otimes \mathbf{C} = g_{\mathbf{C}}[z, z^{-1}]$. Want a bijective correspondence between PERs of LG and $L^{\mathrm{poly}} G$. Abstractly: E is a PER of LG, we have $L^{\mathrm{poly}} g \to LG \to PU(E)$, the first map is exp, the second map is π. For each x we get a 1-parameter subgroup $t \in \mathbf{R} \mapsto \pi(\exp(tx))$.

By Stone's theorem we get $X \in \mathrm{End}(E)$ such that $\exp(tX) = \pi(\exp(tX))$, hence we get a projective representation $\rho\colon L^{\mathrm{poly}} g \to \mathrm{End}(E)$.

Let d denote the infinitesimal generator of $\mathbf{T}$ on E: $d|_{E(n)} = n$ and $R_\theta = \exp(i\theta d)$.

Theorem (Wassermann): Let E be a level l representation of LG. Then

1. $E = \oplus_{n \geq 0} E(n) \subset E$ is preserved by ρ;
2. We can choose lifts in such a way that $[d, \rho(x)] = i\rho(x')$ on E.
3. $[\rho(x), \rho(y)] = \rho([x, y]) + il\omega_{\mathrm{basic}}(x, y)$.

Concretely: $H = L^2(S^1, V)$, $\mathrm{Cliff}(H)$ acts on F_P, the fermionic Fock space. $L^{\mathrm{poly}} V$ acts on $\mathrm{Cliff}(H)$ as follows: $\exp(in\theta) \otimes v \mapsto c(\exp(in\theta)v)$.