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Suppose G is a simple simply-connected Lie group.

For $\widetilde{LG}$ we consider positive energy representation.

There is an action of the group of diffeomorphisms of S^1 on positive energy representations.

We look at the Lie algebra of this group of diffeomorphisms.

We restrict to polynomial elements of this Lie algebra, i.e., elements of the form $\sum_n a_n \exp(in\theta)d/d\theta$. $L_n = i\exp(in\theta)d/d\theta$. Let $\exp(i\theta) = t$. $[L_n, L_m] = (m-n)L_{n+m}$.

The formal disk is the spectrum O of $\mathbf{C}[[t]]$. The formal circe is the formal punctured disk, i.e., the spectrum K of $\mathbf{C}((t))$. We consider vector fields on the formal punctured disk: The derivations of K are $\mathbf{C}((t))d/dt$.

There is a universal one-dimensional extension of the Lie algebra of derivations of K: $0 \to \mathbf{C}K \to \mathrm{Vir} \to \mathrm{Der}\, K \to 0$. Generators are L_n and K, $[L_n, L_m] = (m-n)L_{m+n} + ((m^3-m)/12)[m+n=0]K$. This extension is unique.

A module M over Vir has central charge $c \in \mathbf{C}$ if K acts by c.

If g is a simple finite-dimensional Lie algebra. We have $0 \to \mathbf{C}I \to \hat{g} \to g((t)) \to 0$. There is an action of $\mathrm{Diff}(S^1)$ on LG, which gives an action of Vir on $\hat{g}$. $[L_m.X[n]] = -nX[m+n]$. $X \in g$, $X[n] = X \otimes t^n$.

PERs: $\widetilde{LG} \rtimes \mathbf{T}_r$. If $V = \oplus_n V(n)$, then we require that $V(n) = 0$ for $n < 0$. The element $\partial/\partial\theta$ in the Lie algebra of $\mathbf{T}_r$ gives us L_0. $[d/d\theta, X[m]] = -mX[m]$, hence $v \in V(n)$ implies $X[m] \in V(n-m)$.

Casimir: Fix an inner product in g such that $\langle\theta,\theta\rangle = 2$, where θ is the largest root. If X_j is an orthonormal basis of g, then $\Omega = \sum_j X_j^2$ is in the center of $U(g)$.

In the adjoint representation $\Omega = 2h^\vee 1$. $G = \mathrm{SU}(n)$. In genera: If V is a highest weight representation of weight λ, then $\Omega = \langle\lambda, \lambda+2\rho\rangle 1$. ρ is one half of the sum of positive roots.

Consider $\sum_{j,n} X_j[n]X_j[-n]$. After "renormalization" we have $\sum_{j,n>0}(X_j[n]X_j[-n] + X_j[-n]X_j[n]) + \sum_j X_j^2 = \sum_{j,n} : X_j[n]X_j[n] : + \sum_{j,n>0}[X_j[-n], X_j[n]]$. Here $(: X[m]Y[n] :)$ is $X[m]Y[n]$ if $n \geq m$ and $Y[n]X[m]$ if $n < m$.

Second candidate for Casimir: $\Delta_0 = \sum_{j,n} : X_j[-n]X_j[n] :$.

V is a PER of LG. Δ_0 acts on V. Δ_0 is not in the center.

$[Y[m], \sum_j : X_j[-m]X_j[n] :] = Z_{n+m} - Z_n$ if $m \neq \pm n$ or $Z_{n+m} - Z_n + mY[m]I$ if $m = \pm n$, where $Z_n = \sum_{j,k} \alpha_{j,k} : X_k[n]X_j[m-n] :$ and $[Y, X_j] = \sum_k \alpha_{j,k} X_k$ in g.

$[Y[m], \Delta_0] = m(2YI + \Omega Y)[m] = 2m(l + h^\vee)Y[m] = 2(l+h^\vee)(d/d\theta)Y[m]$.

We can learn 2 things: (1) We can get a central element $\Delta = \Delta_0 + 2(I + h^\vee)(d/d\theta)$. (2) $L_0 = (d/d\theta) = -1/(2(l+h^\vee))\Delta_0$.

$\Delta_m = \sum_{j,n} : X_j[m-n]X_j[n] :$. Then $L_m = -1/(2(l+h^\vee))\Delta_m$.

Theorem: These satisfy the Virasoro relations with central charge $K = l(\dim g)/(l+h^\vee)$.