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Suppose we have a Lie group G. Its irreducible representations form some semisimple category. Tensor product of two irreducibles is a direct some of irreducible with some coefficients: $V_f \otimes V_g = \oplus_h N_{f,g}^h V_h$.

We have the same picture for irreducible representations of the loop group of G at some level l: $H_f \otimes H_g = \oplus_h N_{f,g}^h (-1)^{\sigma_h} H_{h'}$, where σ is in the semidirect product of Λ_0 and S_n.

Representations of $\mathrm{SU}(n)$.

Two principles in representation theory.

1. Studying complex representations of some simply connected Lie group G is the same thing as studying complex representations of its Lie algebra.
2. Studying complex representations of some Lie algebra is the same thing as studying complex representations of $g \otimes_{\mathbf{R}} \mathbf{C}$.

We want to establish a correspondence between highest weight vectors and signature.

Example. $\mathrm{su}(3) \otimes \mathbf{C} = \mathrm{sl}_3(\mathbf{C})$. We want to study the representation of sl_3 on itself, i.e., X acts on v by $[X, v]$.

$h \subset \mathrm{sl}_3$ is the Cartan subalgebra of sl_3, i.e., the diagonal matrices in sl_3. How does this nice subalgebra act on sl_3. Remark: h is killed by h. How does h act on $E_{i,j}$, where $i \neq j$? We have $X(E_{i,j}) = (\alpha_i - \alpha_j)E_{i,j}$. (Here α_i is the ith diagonal entry of X.)

Define $L_i \in h^\vee = \mathrm{Hom}(h, \mathbf{C})$ by $L_i \colon X \mapsto X_{i,i}$. Then $X(E_{i,j}) = (L_i - L_j)(X)E_{i,j}$.

Claim: An off-diagonal matrix $E_{i,j}$ of weight β will take some v in the eigenspace α, where $\alpha \in h^\vee$ and take it to $v_{\alpha+\beta} \in g_{\alpha+\beta}$.

The upshot: Upper triangular matrices raise vectors, lower triangular matrices lower vectors.

Observation: There is some $\alpha \in h^\vee$ such that g_α is in the kernel of all raising operators. Definition: Such an α is called the highest weight of a representation. Example: $L_1 - L_3$ is the highest weight of the adjoint representation of sl_3.

The orbit of the highest weight vector under all lowering operators covers the entire (irreducible) representation.

Signatures: A signature (positive weight) is an element $f \in \mathbf{Z}^n$ such that $f_1 \geq \cdots \geq f_n \geq 0$.

Question: Is there an irreducible representation of $\mathrm{sl}_n(\mathbf{C})$ such that the highest weight vector has weight $\sum f_i L_i$. Answer: Yes.

We construct Young diagrams from signatures in the obvious way.

Pieri rule: The tensor product of V_f by V of a column consisting of k cells is the direct sum of all V_g where g differs by f by adding k boxes to f such that no two boxes go to the same row.

Remark: When looking at representations of $L\widetilde{\mathrm{SU}(n)}$ signatures need to be permissible, i.e., $f_1 - f_n \leq l$.

Theorem: (Verlinde formula.) If $V_f \otimes V_g = \oplus_h N_{f,g}^h V_h$, then $H_f \otimes H_g = \oplus_h N_{f,g}^h \operatorname{sign}(\sigma_n) H_{h'}$.

Action of the affine Weyl group. Definition: $\Lambda_0\{(N+l)(m_i) \mid \sum_i m_i = 0\}$. S_n is the symmetric group on n letters. Definition; The affine Weyl group is the semidirect product of Λ_0 and S_n.