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Hypergeometric functions.

The hypergeometric equation is a second order ODE with three regular singular points 0, 1, and ∞: $z(1-z)f'' + (c-(1+a+b)z)f' - abf = 0$. We can restate it as $F'(z) = (A/z + B/(1-z))F(z)$. Here $A = \begin{pmatrix} 0 & 1 \\ 0 & 1-c \end{pmatrix}$, $B = \begin{pmatrix} 0 & 0 \\ -ab & 1+a+b-c \end{pmatrix}$, $F(z) = \begin{pmatrix} f(z) \\ zf'(z) \end{pmatrix}$.

$G(a,b;c;z) = \sum_{n\geq 0}(a^{\bar{n}}b^{\bar{n}}/c^{\bar{n}})z^n/n!$.

$f'(z) = Pf/z + Qf/(1-z)$.

$f\colon \mathbf{C} \to V$, $P, Q \in \mathrm{End}(V)$. P has eigenvalues λ_i (they are different) and eigenvectors ξ_i. Q is a nonzero multiple of a rank one idempotent. $Q^2 = \delta Q$, $\delta \neq 0$. Hence $\mathrm{tr}(Q) = \delta$. Rank 1: There is $\phi \in V^*$ and $v \in V$ so that $Q(x) = \phi(x)v$, $\phi(v) = \delta$.

Q is in general position with respect to P. $v = \sum \delta_i \xi_i$, where $\delta_i \neq 0$.

Choose eigenvectors so that $\phi(\xi_i) = 1$. Let $R = Q - P$ and suppose R satisfies the same conditions as P with respect to Q. Let $(\zeta_j, -\mu_j)$ be the normalized eigenvectors/eigenvalues. $f_i(z) = \sum_n \xi_{i,n} z^{\lambda_i + n}$. $\xi_{i,0} = \xi_i$. Converges in $\{z \mid |z| < 1 \wedge z \notin [0,1)\}$. $g_j(z) = \sum_n \zeta_{j,n} z^{\mu_j - n}$. $\xi_{j,0} = \zeta_j$. Extend analytically and complete. $f_i(z) = \sum_j c_{i,j} g_j(z)$. Goal: compute $c_{i,j}$.

$c_{i,j} = \exp(i\pi(\lambda_i - \mu_j)) \prod_{k\neq i} \Gamma(\lambda_i - \lambda_k + 1) \prod_{l \neq j} \Gamma(\mu_j - \mu_l) / \prod_{l\neq j} \Gamma(\lambda_i - \mu_l + 1) \prod_{k \neq i} \Gamma(\mu_j - \lambda_k)$.

Fact: The transport matrix depends only on the eigenvalues of P and $P - Q$. This dependence is holomorphic.

Fact: $\delta_1 \sigma \in S_n$. $c_{i,j}(\lambda_1, \ldots, \lambda_n, \mu_1, \ldots, \mu_n) = c_{\sigma i, \tau j}(\lambda_{\sigma_1}, \ldots, \lambda_{\sigma_n}, \mu_{\tau_1}, \ldots, \mu_{\tau_n})$.

Look at $\phi(f_i(z))$ because $\phi(f_i(z)) = \sum_j c_{i,j}\phi(g_j(z))$. $f_1(z) = \sum \xi_{i,n} z^{\lambda_1 + n}$.

$\sum_{n\geq 0}(n+\lambda_1)\xi_{i,n}z^n = \sum_{n\geq 0} P\xi_{1,n}z^n + Q(1 + z + z^2 + \cdots)\sum_{n \geq 0}\xi_{i,n}z^n$.

$(n + \lambda_1 - P)\xi_{1,n} = Q(\xi_{1,0} + \cdots + \xi_{1,n-1})$ $\alpha_{1,n} = \phi(\xi_{1,0} + \cdots + \xi_{1,n})$.

$\alpha_{1,0} = 1$.

$\alpha_{1,n} = \prod_{1 \leq j \leq N} \prod_{1 \leq m \leq n} (m + \lambda_i - \mu_j)/(m + \lambda_i - \lambda_j)$.

$\phi(f_1(z))/(z^{\lambda_1}(1-z)) = \sum_{n \geq 0} \alpha_{1,n} z^n = \sum_{n\geq 0} z^n \prod\prod$.

Restrict the λ_i and μ_j to real numbers, $\lambda_i > \lambda_{i+1}$ and $\lambda_1 + 1 > \mu_j > \lambda_j$ for all j.

$\Gamma(a)\Gamma(b)/\Gamma(a+b) = \int_0^1 t^{a-1}(1-t)^{b-1}dt$ for $a, b > 0$.

$\phi(f_1(z)) = (1-z)z^{\lambda_1} K \int \cdots \int (1 - zt_2 \cdots t_n)^{\mu_1 - \lambda_1 - 1} \prod_{j\neq 1} t_j^{\lambda_1 - \mu_j}(1 - t_j)^{\mu_j - \lambda_j - 1} dt_j$. $K = \prod_{j \neq 1}\Gamma(\lambda_1 - \lambda_j + 1)/\Gamma(\lambda_1 - \mu_j + 1)\Gamma(\mu_j - \lambda_j) \approx K \exp(i\pi\lambda_1)|z|^{\mu_1} \prod_{j\neq 1} \int_{[0,1]} t_j$.

$\phi(g;(z)) \approx |z|^{\mu_j} \exp(\pi i \mu_j)$. $g_j(z) = \sum \zeta_{j,n} z^{\mu_j - n}$. $\phi(f,(z)) \approx c_{1,1}|z|^{\mu_1}\exp(\pi i \mu_1)$.