André Henriques.

What is conformal field theory?

1. Von Neumann algebras ("conformal nets").
2. Vertex algebras.
3. Segal CFT.

Examples for (1):

1. Loop group nets.
2. Dirac fermions.

G is a compact simply connected Lie group. In Wassermann's paper $G = \mathrm{SU}(n)$. $LG = C^\infty(S^1, G)$. $\widetilde{LG}$ is a central extension of LG by S^1. (The set of extensions is isomorphic to integers.) The extension is determined by an integer l (the level). We assume $l \geq 0$, because otherwise we cannot find an interesting representation H_0 of $\widetilde{LG}$.

Given $I \subset S^1$, consider $L_I G = \{\gamma\colon S^1 \to G \mid \forall z \notin I\colon \gamma(z) = 1\}$. We can form a pullback $\widetilde{L_I G}$. $\widetilde{L_I G}$ acts on H_0.

Define $A(I)$ to be the von Neumann algebra generated by $\widetilde{L_I G}$. $I \mapsto A(I)$ is a conformal net, together with the following data. The group $\mathrm{SL}_2(\mathbf{R})$ also acts on H_0. The vacuum vector $\Omega \in H_0$ is invariant under this action. Summary: For every G and for every nonnegative integer l we have a conformal net.

Let's forget the Hilbert space of a conformal net. For every inclusion of intervals we have an inclusion of the corresponding von Neumann algebras. A representation of a conformal net is a family of maps $A(I) \to B(H_\lambda)$ compatible with inclusions of algebras.

Representations of loop group conformal nets of level l are in bijective correspondence with projective level l positive energy representations of LG.

We look at Hilbert spaces with actions of two von Neumann algebras. We have a nice tensor product of such bimodules, the Connes fusion.

If two intervals on a circle are disjoint, then the corresponding algebras commute as subalgebras of $B(H_\lambda)$. If we take two disjoint intervals that are complements of each other. If we have an orientation reversing isomorphism between two intervals, then we have an isomorphism of corresponding von Neumann algebras, if we reverse the order of multiplication on one of them. Thus we obtan a bimodule over two von Neumann algebras corresonding to two semicircles.

Now we consider tensor products of representations: $H_\lambda \otimes H_\mu = \oplus_\nu N^\nu_{\lambda,\mu} H_\nu$. What are these coefficients N?

Answer: The ring of representations of LG at level l is a quotient of the usual ring of finite-dimensional representations of G.

Theorem: For any conformal net that is finite in a certain sense its category of representations is modular.