FOCK REPRESENTATIONS

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In this note we use Clifford algebras and their Fock representations to build representations of $LU(N)$ and $LSU(N)$. Let H be the Hilbert space of square-integrable sections of the trivial rank N complex vector bundle over S^1 and $P : H \to H$ the projection onto the space spanned by non-negative Fourier modes. To this data we can associate the Fermionic Fock space $\mathfrak{F}_P$. $\mathfrak{F}_P$ is a representation of level 1 of $LU(N)$. It turns out that all positive energy representations $LSU(N)$ at level l are contained in the Fermionic representation $\mathfrak{F}_P^{\otimes l}$.

1. Clifford Algebras and Fock Representations

1.1. **The Clifford Algebra.** Let $(H, \langle\ ,\ \rangle)$ be a complex Hilbert space, the complex Clifford algebra $C(H)$ is the unital $*$-algebra generated by a complex linear map $f \mapsto c(f)$ for $f \in H$ satisfying the anticommutation relations

$$c(f)c(g) + c(g)c(f) = 0$$

and

$$c(f)c(g)^* + c(g)^*c(f) = \langle f, g\rangle.$$

Note that we can realize $C(H)$ explicitly by

$$C(H) = T(H)/(f \otimes f - \langle f, f\rangle).$$

The Clifford algebra has a natural action on ΛH given by $\pi(c(f))\omega = f \wedge \omega$, the complex wave representation. Let $\Omega = 1 \in \Lambda^0 H$ be the vacuum vector, which is cyclic. The annhilation action $a(f) = c(f)^*$ is given by annhilating Ω and on decomposables by

$$a(f)(\omega_0 \wedge \cdots \wedge \omega_n) = \sum_{j=0}^{n} (-1)^j \langle \omega_j, f\rangle \omega_0 \wedge \cdots \wedge \widehat{\omega_j} \wedge \cdots \wedge \omega_n.$$

Annhilation and creation really are adjoint with respect to the the inner product

$$\langle \omega_0 \wedge \cdots \wedge \omega_n, \eta_0 \wedge \cdots \wedge \eta_n\rangle = \mathrm{Det}[\langle \omega_i, \eta_j\rangle].$$

Proposition 1.1. *The wave representation is irreducible.*

Proof. Let $T \in \mathrm{End}(\Lambda H)$ commuting with all $a(f)$'s, so for each $f \in H$

$$a(f)T\Omega = Ta(f)\Omega = 0,$$

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hence $T\Omega = \lambda\Omega$ for some $\lambda \in \mathbb{C}$ as

$$\cap_{f\in H} \ker a(f) = \Omega\mathbb{C}.$$

Indeed if $\zeta \in \cap \ker a(f)$ then

$$\langle \zeta, f_0 \wedge \cdots \wedge f_m \rangle = \langle a(f_0)\zeta, f_1 \wedge \cdots \wedge f_m \rangle = 0,$$

and by linearity ζ is orthogonal to all elements $\bigoplus_{n>0} \Lambda^n(H)$ and hence lies in $\Lambda^0(H) = \Omega\mathbb{C}$. Now if T also commutes with all $c(f)$'s, then $T = \lambda I$ as Ω is cyclic for the $c(f)$'s. □

1.2. **Unitary Structures.** Now, let $(V, (,))$ be a real Hilbert space of dimension other than odd (i.e. even or infinite). A *unitary structure* on V is $J \in O(V)$ such that $J^2 = -I$. We use J to make V into a complex vector space $(iv = J(v))$ equipped with a Hermitian inner product which we will denote by $(V_J, \langle\ ,\ \rangle)$ where

$$\langle v, w\rangle = (v, w) + i(v, J(w)).$$

We can use unitary structures to get more irreducible representations of Clifford algebras. Given a unitary structure J, define

$$P_J = \frac{1}{2}(I - iJ) \in \mathrm{End}(V \otimes_{\mathbb{R}} \mathbb{C}).$$

Let H be the Hilbert space $V \otimes_{\mathbb{R}} \mathbb{C}$ with inner product

$$\langle x \otimes \mu, y \otimes \nu\rangle = (x, y)\mu\overline{\nu}.$$

As $J^2 = -I$, we deduce that P_J is a projection operator. Denote by $F_J = P_J H$, which is also the $+i$ eigenspace of J.

The above discussion is symmetric in the sense that if P is a projection on a $\mathbb{C}$-vector space such that $P + \Sigma P \Sigma = I$, where Σ denotes complex conjugation, then

$$J = i(2P - I)$$

defines a unitary structure on V.

Given a P_J as above define the *fermionic Fock space* $\mathfrak{F}_P = \Lambda P_J H \widehat{\otimes} \Lambda(P_J^{\perp} H)^*$. $C(V_J)$ acts irreducibly by $\pi_J(c(f)) = c(Pf) + c((P^{\perp} f)^*)^*$. In terms of F_J we define $\mathfrak{F}(F_J) = \Lambda(\overline{F}_J)$ and complete with respect to the induced inner product $\overline{F}_J \subset V$. $\overline{v} \in \overline{L}$ acts via creation and $v \in L$ acts via annhilation. These two notions agree, despite our insistence on different notation.

I. Segal-Shale Equivalence Criterion. *If J and K are unitary structures on V, then the following are equivalent:*

(1) *the Fock representations π_J and π_K are unitarily equivalent;*
(2) *the difference $P_K - P_J$ is a Hilbert-Schmidt operator;*

(3) *the composite linear operator*

$$F_K \subset V \to \overline{F}_J$$

is Hilbert-Schmidt.

Proof.
(2) $\Rightarrow$ (3): this is immediate as the operator can be identified with $(P_K - P_J)|_{F_K}$.
(3) $\Rightarrow$ (2): the claim follows as

$$\begin{aligned} P_K - P_J &= (I - P_J)P_K - P_J(I - P_K) \\ &= (I - P_J)P_K - \Sigma(I - P_J)P_K\Sigma \end{aligned}$$

and $(I - P_J)P_K$ is zero on $\overline{F}_K \overset{\text{def}}{=} \Sigma F_K$ and restricts to F_K as the operator $F_K \to \overline{F}_J$ as in (3).
(2) $\Rightarrow$ (1) (the converse is also true, but we don't give the proof):

- If the representations are finite then they are equivalent to ΛH, so we may assume that they are infinite dimensional.
- $T \overset{def}{=} (P_K - P_J)^2$ is compact, so by the spectral theorem
$$H = \bigoplus_{\lambda \geq 0} H_\lambda$$
and $P_K = P_J$ on H_0.
- T commutes with both P_K and P_J, so the H_λ are invariant under P_K and P_J. We can therefore further decompose H into
$$H = \bigoplus_j V_j$$
where V_j are finite dimensional irreducible submodules for P_K and P_J which are also eigenspaces for T.
- For any j, P_K and P_J (and I) generate $\mathrm{End}(V_j)$, so $\dim \mathrm{End}(V_j) \leq 4$ and therefore $\dim V_j = 1$ or 2.
- We can choose an orthonormal basis $(e_i)_{i \geq -a}$ for $P_K^\perp H$ with each e_i in some V_j and
$$P_J^\perp e_{-1} = \cdots = P_J^\perp e_{-a} = 0 \text{ and } P_J^\perp e_i \neq 0$$
for $i \geq 0$. We complete to an orthonormal basis of H by adding vectors from the V_j's.
- Let $(f_l)_{k \geq -b}$ be an orthonormal basis for $P_J^\perp H$ such that $f_l \in V_j \ni e_l$ if $l \geq 0$ and $\langle e_l, f_l \rangle > 0$.
- If V_j is a λ_i-eigenspace for T and e_l and $f_l \in V_j$, then $\langle e_l, f_l \rangle = \sqrt{1 - \lambda_i}$.
- $\|P_K - P_J\|_2^2 = \mathrm{Tr}T = a + b + 2\sum \lambda_i$, so in particular $\sum \lambda_i < \infty$.

- We now build a representation of $C(V)$ that intertwines the representations π_K and π_J. Let $\mathcal{H}$ be the Hilbert space with orthonormal basis given by symbols $e_{i_1} \wedge e_{i_2} \wedge \cdots$ where $i_1 < i_2 < \cdots$ and $i_{k+1} = i_k + 1$ for k large. Then $\pi(c(f)) = f\wedge$ yields a representation of $C(V)$.
- Define the cyclic vector $\xi \in \mathcal{H}$ by
$$\xi = e_{-a} \wedge e_{-a+1} \wedge \cdots .$$
- $\langle \pi(a)\xi, \xi\rangle = \langle \pi_K(a)\Omega_K, \Omega_K\rangle$ and $U(\pi_K(a)\Omega_K) \overset{def}{=} \pi(a)\xi$ defines a unitary from $\mathfrak{F}_{P_K}$ onto $\mathcal{H}$ such that $\pi(a) = U\pi_K(a)U^*$ for some unitary $U \in U(\mathfrak{F}_{P_K})$.
- To complete the proof it is enough to find $\eta \in \mathcal{H}$ such that
$$\langle \pi(a)\eta, \eta\rangle = \langle \pi_J(a)\Omega_J, \Omega_J\rangle.$$
- Define
$$\eta_N = f_{-b} \wedge \cdots \wedge f_{-1} \wedge f_0 \wedge \cdots f_N \wedge e_{N+1} \wedge e_{N+2} \wedge \cdots .$$
It is clear that for $a \in C(V)$ there exists an N large enough (depending on a) such that
$$\langle \pi(a)\eta_N, \eta_N\rangle = \langle \pi_J(a)\Omega_J, \Omega_J\rangle.$$
Hence we need to show that the sequence $\{\eta_N\}$ has a limit.
- $\{\eta_N\}$ is a Cauchy sequence. Indeed,
$$\langle \eta_N, \eta_M\rangle = \prod_{i=M+1}^{N} (e_i, f_i) = \prod_{i=M+1}^{N} \sqrt{1-\lambda_i}$$
and as $\sum \lambda_i < \infty$, $\mathrm{Re}\langle \eta_N, \eta_M\rangle \to 1$ as $M \le N \to \infty$.

□

2. Implementation and the Basic Representation

Let $u \in U(V_J)$, then u yields an automorphism of $C(V)$ via $c(f) \mapsto c(uf)$. An automorphism is said to be *implemented* in π_J (or $\mathfrak{F}_P$) if $\pi_J(c(uf)) = U\pi_J(c(f))U^*$ for some unitary $U \in U(\mathfrak{F}_P)$ unique up to a phase (i.e. $\phi \in \mathbb{C}$).

Proposition 2.1. *u is implemented in $\mathfrak{F}_P$ if $[u, P]$ is Hilbert-Schmidt.*

Proof. Let $Q = u^*P_Ju$, for J a unitary structure and let K be the unitary structure corresponding to Q. Then $\pi_J(c(uf)) = \pi_K(c(f)) = U\pi_J(c(f))U^*$, where the last equality follows from the equivalence criterion. □

Define the *restricted unitary group* $U_P(V_J) = \{u \in U(V_J) : [u, P] \text{ Hilbert-Schmidt}\}$, it is a topological group under the strong operator topology and the Hilbert-Schmidt norm. By the above corollary, there is a homomorphism $\pi : U_P(V_J) \to PU(\mathfrak{F}_P)$, called the *basic* projective representation.

Lemma 2.2. *The basic representation is continuous.*

Recall that it is enough to verify the above lemma at the identity, i.e. if $u_n \xrightarrow{s} I$ and $\|[u_n, P]\|_2 \to 0$, then there exists lifts $U_n \in U(\mathfrak{F}_P)$ of $\pi(u_n)$ such that $U_n \xrightarrow{s} I$.

Note that if $[u, P] = 0$, then u is canonically implemented in $\mathfrak{F}_P$ (i.e. we actually have a unitary action) and we refer to this as *canonical quantization*. Similarly if $uPu^* = I - P$ then u is canonically implemented by a conjugate-linear isometry in $\mathfrak{F}_P$ again called canonical quantization.

3. Relation to CFT

In [2], Segal associates to a Lie group G and level $\ell \in H^4(BG; \mathbb{Z})$ a weakly conformal field theory. We now describe this construction.

We need to associate to each closed one manifold equipped with a label a vector space and to each surface with boundary a finite dimensional vector space built from the boundary vector spaces. Labels in this context correspond to positive energy representations of LG at level ℓ. Let E denote the field theory, then $E(S^1, \rho) = V_\rho$.

Let $Y_0 \xrightarrow{\Sigma} Y_1$ be a bordism with boundary components equipped with labels. Let $N(Y_i)$ be a tubular neighborhood in Σ, then we have a restriction map

$$\mathrm{Hol}(\Sigma, G_{\mathbb{C}}) \to \mathrm{Hol}(N(Y_0), G_{\mathbb{C}}) \times \mathrm{Hol}(N(Y_1), G_{\mathbb{C}}).$$

We have a natural (projective) action of $\mathrm{Map}(Y_0, G) \times \mathrm{Map}(Y_1, G)$ on the vector space V_Σ, where

$$V_\Sigma = \bigotimes_{\rho \in \mathrm{Labels}(Y_0)} V_\rho^\vee \otimes \bigotimes_{\eta \in \mathrm{Labels}(Y_1)} V_\eta.$$

Via the two maps $\mathrm{Hol}(N(Y_i), G_{\mathbb{C}}) \to \mathrm{Map}(Y_i, G)$, we get an action of $\mathrm{Hol}(\Sigma, G_{\mathbb{C}})$ on V_Σ, then define

$$E(\Sigma) = V(\Sigma)^{\mathrm{Hol}(\Sigma, G_{\mathbb{C}})}.$$

It turns out that this fixed subspace is finite dimensional and further there is an $n \in \mathbb{Z}$, such that

$$E(\Sigma) = E'(\Sigma) \otimes \mathrm{Det}^{\otimes n}(\Sigma).$$

The $E'(\Sigma)$ corresponds to a three dimensional TFT, namely Chern-Simons.

It is of note that in order to see the fusion of PERs on the CFT side, one must be able to extend Chern-Simons to points.

3.1. The Action of $LU(N)$.

Define $H = L^2(S^1) \otimes \mathbb{C}^N$, and $P : H \to H_{\geq 0}$ the projection onto the Hardy space. For $f \in C^\infty(S^1, \mathrm{End}(\mathbb{C}^N))$, let $m(f)$ denote the corresponding multiplication operator. One can check that

$$\|[P, m(f)]\|_2 \leq \|f'\|_2.$$

So $LU(N)$ satisfies the quantization criterion and we get a projective representation of $LU(N)$ on $\mathfrak{F}_P$ which is continuous for the subspace topology $LU(N) \subset C^\infty(S^1, \mathrm{End}(V))$. Similarly, there is a continuous projective representation of $LSU(N)$ on $\mathfrak{F}_P$.

Let $G = U(N)$ and $1 = \ell \in H^4(BU(N), \mathbb{Z}) \cong \mathbb{Z}$. Let ρ be such that $E(S^1, \rho) = \mathfrak{F}_P$. Consider the disk Σ as a bordism from $\emptyset$ to (S^1, ρ), we then have

$$V_\rho^{\mathrm{Hol}(\Sigma, GL(N))} = V_\rho^{H_{\geq 0}} = \Omega\mathbb{C},$$

where $\Omega = 1 \in \Lambda^0(PH) \subset \mathfrak{F}_P$ is the vacuum vector.

3.2. **The** Rot S^1 **Action.** The rotation group Rot S^1 acts on $LU(N)$ (or any LG) by $(r_\alpha f)(\theta) = f(\theta + \alpha)$. Similarly, Rot S^1 acts in a unitary fashion on $H = L^2(S^1) \otimes \mathbb{C}^N$ which leaves $H_{\geq 0}$ invariant, hence this action is canonically quantized. As a result we get a projective representation of $LU(N) \rtimes$ Rot S^1 on $\mathfrak{F}_P$ which restricts to an honest representation on Rot S^1. The spectral decomposition of this Rot S^1 action gives the *energy* grading.

3.3. **The** $\mathrm{Diff}(S^1)$ **Action.** Consider the subgroup of $\mathrm{Diff}(S^1)$ which extend to the disk in a way that preserves the conformal structure. This group (or rather its double cover) is $SU_\pm(1,1)$ which can be described explicitly by

$$SU_\pm(1,1) = \left\{ \begin{pmatrix} \alpha & \beta \\ \overline{\beta} & \overline{\alpha} \end{pmatrix} : |\alpha|^2 - |\beta|^2 = \pm 1 \right\}.$$

Let $SU_+(1,1)$ denote the elements which preserve orientation i.e. have determinant 1. Note that $SU_-(1,1)$ is a coset of $SU_+(1,1)$ with representative $F = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$. The action of $SU_\pm(1,1)$ on S^1 is given by

$$g(z) = \frac{\alpha z + \beta}{\overline{\beta} z + \overline{\alpha}},$$

which leads to a unitary action on H via

$$(V_g \cdot f)(z) = \frac{f(g^{-1}(z))}{(\alpha - \overline{\beta} z)}.$$

For $|z| < 1$ and $|\alpha| > |\beta|$ $(\alpha - \overline{\beta} z)^{-1}$ is holomorphic, so for each $g \in SU_+(1,1)$, V_g commutes with the projection P and hence the action is canonically quantized. Note that $(V_F \cdot f)(z) = z^{-1} f(z^{-1})$, so $FPF = I - P$ and hence F is canonically implemented in $\mathfrak{F}_P$ by a conjugate linear isometry fixing the vacuum vector. We thus have an orthogonal representation of $SU_\pm(1,1)$ for the underlying real inner product on $\mathfrak{F}_P$, with $SU_+(1,1)$ preserving the complex structure and $SU_-(1,1)$ reversing it. The same is true in $\mathfrak{F}_P^{\otimes \ell}$.

3.4. **The Charge Grading.** Consider the constant loops $U(1) \subset LU(1)$ sitting inside of $LU(N)$ via the diagonal embedding. This action is given by multiplication by z on H, the action is canonically quantized and we let U_z denote the operator on $\mathfrak{F}_p$ corresponding to $z \in U(1)$. Note that this $U(1)$ action gives the usual grading of the (non-completed) exterior algebra ΛPH and the inverted grading on $\Lambda(P^\perp H)^*$, the total grading on $\mathfrak{F}_P$ is called the *charge* grading, i.e. $\omega \in \Lambda^p PH \otimes \Lambda^q(P^\perp H)^*$ has charge $p - q$. The $\mathbb{Z}/2$-grading on $\mathfrak{F}_p$ is given by U_{-1} eigenspaces.

Lemma 3.1. *Let $z \in U(1)$, then for all $g \in LSU(N)$, $\pi(g)U_z\pi(g)^* = U_z$. That is, the $LSU(N)$ action is compatible with the charge grading.*

Corollary 3.2. *The operator $\pi(g)$ is even (i.e. commutes with U_{-1}) for all $g \in LSU(N)$.*

4. Functoriality of the Fock Representation

Note the notation has changed, in what follows $F(L)$ represents $\mathfrak{F}_P$, where $L = PV$. The language of generalized lagrangians is more amenable to what follows and hence the change in notation.

Symplectic reduction is a quotient construction for symplectic vector spaces. Let (V, ω) be a symplectic vector space, i.e. ω is a non-degenerate, skew-symmetric, bilinear form. For a subspace $U \subset V$, the *annhilator* $U^\perp$ is defined by

$$U^\perp := \{v \in V : \omega(v, u) = 0 \text{ for all } u \in U\}.$$

A subspace $U \subset V$ is *isotropic* if $U \subseteq U^\perp$. Given an isotropic subspace $U \subset V$ we produce a new symplectic vector space (W, η) called the *symplectic reduction* of (V, ω). (W, η) is defined as

$$W := U^\perp/U \text{ with symplectic form } \eta([u_1], [u_2]) := \omega(u_1, u_2).$$

It is easy to see that $\dim W = \dim V - 2\dim U$. Further, if $L \subset V$ is a Lagrangian, then we get a Lagrangian L^{red}, where

$$L^{\text{red}} := \left(L \cap U^\perp\right)/(L \cap U).$$

We would like to view the assignments $V \mapsto C(V)$ and $L \mapsto F(L)$ as a functor. The objects of the domain category are complex Hilbert spaces with involutions and morphisms from V_1 to V_2 are Lagrangian subspaces of $V_2 \oplus -V_1$. Given two morphisms L_1 and L_2 which we visualize as

$$V_1 \xrightarrow{L_1} V_2 \xrightarrow{L_2} V_3$$

we want to form their composition which is a Lagrangian of $V_3 \oplus -V_1$. This is accomplished via symplectic reduction of the Lagrangian

$$L = L_2 \oplus L_1 \subset V_3 \oplus -V_2 \oplus V_2 \oplus -V_1$$

with regards to the isotropic subspace

$$U = \{(0, v_2, v_2, 0) | v_2 \in V_2\} \subset V_3 \oplus -V_2 \oplus V_2 \oplus -V_1.$$

Indeed this yields the desired result as $U^{\perp}/U \cong V_3 \oplus -V_1$.

The range category of our potential functor is that of $\mathbb{Z}/2$-graded algebras. Explicitly, the objects of this category are $\mathbb{Z}/2$-graded algebras and the morphisms are pointed, graded bimodules. The composition of a pointed B-A-bimodule (M, m_0) and a pointed C-B-bimodule (N, n_0) is the pointed C-A-bimodule $(N \otimes_B M, n_0 \otimes m_0)$. If $C(V)$ generates a Type I von Neumann algebra in $B(F(L))$, then the Clifford algebra and Fock space construction is a lax functor (see [3]). In the case that $C(V)$ is not of Type I, we need to use Connes' Fusion.

References

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