TRANSPORT FORMULA, CONNES FUSION WITH THE VECTOR REPRESENTATION

SPEAKER: ANDRÉ HENRIQUES
TYPIST: EMILY PETERS

ABSTRACT. Notes from the "Conformal Field Theory and Operator Algebras workshop," August 2010, Oregon.

Recall a formula from Scott's talk, of the form

$$\phi \circ \phi = \sum c\phi \circ \phi$$

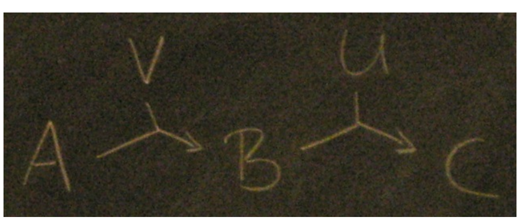
with indices of the ϕs switched around.

Here's a finite-dimensional analogy:

We need 5 irreps of $SU(N)$, A, B, C, U, V. U and V are miniscule, meaning $V_{[k]} = \wedge^k \mathbb{C}^N$ for some k. This means $\mathrm{Hom}(- \otimes V, -)$ is at most one dimensional (recall Pieri rule from earlier today).

Once and for all, fix a basis vector $\phi^V_{X,Y}$ of $\mathrm{Hom}(X \otimes V, Y)$.

Consider



It is an element in

$$\mathrm{Hom}(A \otimes V \otimes U, C) = \mathrm{Hom}(A \otimes U \otimes V, C)$$

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(this uses the fact that the cat of $SU(N)$ reps has a symmetric braiding). And

$$\mathrm{Hom}(A \otimes U \otimes V, C) = \bigoplus_{B'} \mathrm{Hom}(A \otimes U, B') \otimes \mathrm{Hom}(B' \otimes V, C)$$

Assume our index set contains only those B' which make a non-zero contribution.

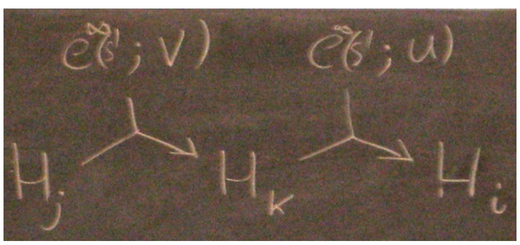
Say our original element is $\phi^U_{CB} \circ \phi^V_{BA}$; elements in the direct sum are of the form $\phi^V_{B'A} \otimes \phi^V_{CB'}$.

If I'm given elements $v \in V$, $u \in U$, I have

$$\phi^U_{CB}(u)\phi^V_{BA}(v) = \sum_{B'} c_{BB'} \phi^V_{CB'}(v)\phi^U_{B'A}(u)$$

which is an equation in $\mathrm{Hom}(A, C)$.

We now do the same thing for primary fields. We have H_i, H_j, H_k and instead of U, V we'll have C^∞ functions to U or V.



Given $a \in C^\infty(S^1, U)$ and $b \in C^\infty(S^1, V)$ with disjoint support (and disjoint from $\{1\}$), we get

$$\phi^U_{ik}(a)\phi^V_{kj}(b) = \sum_{k'} c_{kk'} \phi^V_{ik'}(b)\phi^U_{k'j}(a).$$

What's important is that we have complete control over which $c_{kk'}$ are and aren't zero, since we have a formula of the form

$$c_{kk'} = \frac{\prod \Gamma(\cdots) \prod \Gamma(\cdots)}{\prod \Gamma(\cdots) \prod \Gamma(\cdots)}$$

Now, if a and b were delta functions, this would be a solution of the KZ-equation. As it is, it's a smeared solution to the KZ-equation.

Goal: we want to understand $H_\Box \boxtimes H_f$; we expect it to be the direct sum over g obtained by adding one box to f. This isn't going to be quite enough; as $H_\Box$ doesn't generated the fusion ring, we also need to understand $H_{[k]} \boxtimes H_f$.

This is done with similar techniques and is twice as technical; Wasserman does it, but we won't.

$H_\Box \boxtimes H_f$ is a completion of $\mathrm{Hom}_{L_{I'}\tilde{G}}(H_0, H_\Box) \otimes H_0 \otimes \mathrm{Hom}_{L_I\tilde{G}}(H_0, H_f)$. This (minus the H_0) is what Yoh talked about. It's also a completion of

$$\mathrm{Hom}_{L_{I'}\tilde{G}}(H_0, H_\Box) \otimes H_f$$

– this is the nonsymmetric version of Connes fusion. This is where we're going to work.

Given $a \in L^2(I, V)$ ($V = \mathbb{C}^N = V_\Box$)) , you get $a_{\Box 0} = \phi^\Box_{\Box 0}(a) \in \mathrm{Hom}_{L_{I'}\tilde{G}}(H_0, H_\Box)$

Given $h \in L_I\tilde{G}$ you get $\pi_\Box(h) \in \mathrm{Hom}_{L_{I'}\tilde{G}}(H_\Box, H_\Box)$

Elements of the form $x = \sum_n \pi_\Box(h^n) a^n_{\Box\Box}$ are dense in $Hom(H_n, H_\Box)$.

Goal: We simplify our lives and look just at $x = a_{\Box 0}$, $y \in H_f$. We want to compute the norm of $x \otimes y \in H_\Box \boxtimes H_f$.

By definition,

$$\begin{aligned} \|x \otimes y\|^2 &= \langle x^* x y, y\rangle \\ &= \langle \pi_f(x^* x) y, y\rangle \end{aligned}$$

(Note this middle term doesn't really make sense, but we make sense of it by:) $x^* x \in \mathrm{Hom}_{\mathcal{A}(I')}(H_0, H_0) = \mathcal{A}(I)$ by Haag duality. This $\mathcal{A}(I)$ acts everywhere

Theorem 0.1. *Technical theorem (transport formula) [W]:*

$\pi_f(a^*_{\Box 0} a_{\Box 0}) \sum_{g=f+\Box \text{ permissible}} \lambda_g a^*_{gf} a_{gf}$ *with all* $\lambda_g > 0$.

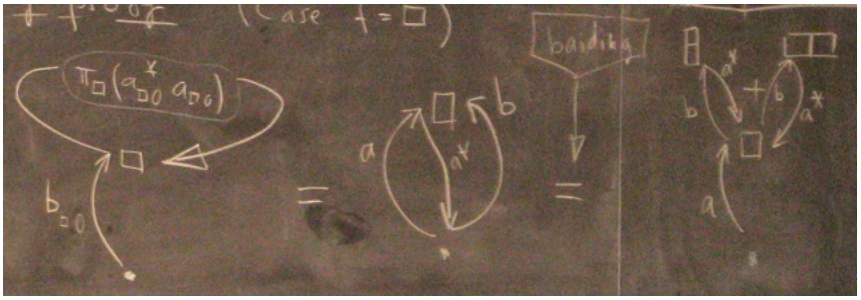
Showing that $\lambda_g \geq 0$ isn't bad; showing they're not $= 0$ is the hard part.

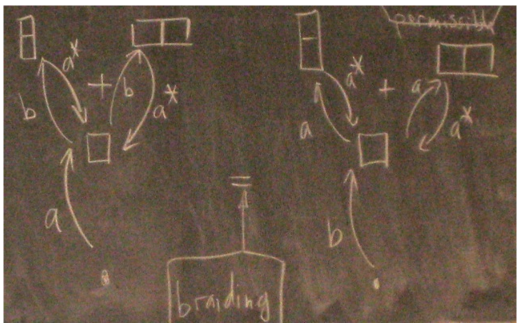
Proof. Sketch:

$f = 0$ is easy, equality on both sides.

What about $f = \Box$? $\pi_\Box(a^*_{\Box 0} a_{\Box 0})$ takes $H_\Box$ back to itself. Let's precompose now with a map from the trivial to the box rep: $b_{\Box 0}$.

First, we rewrite it with a^* and a separated. By the braiding we have the second and third equalities.

 and



This is true for all b. By denseness it follows more generality. Non-zeroness follows from non-zeroness of braiding things, which we already knew. $\square$

Now back to our computation:

$$\begin{aligned}
\|x \otimes y\|^2 &= \langle x^* x y, y\rangle \\
&= \langle \pi_f(x^* x) y, y\rangle \\
&= \sum_{g = f + \square} \lambda_g \langle a^*_{gf} a_{gf} y, y\rangle \\
&= \sum_{g = f + \square} \langle \sqrt{\lambda_g} a_{gf} y, \sqrt{\lambda_g} a_{gf} y\rangle
\end{aligned}$$

Conclusion. $H_\square \boxtimes H_f \to \bigoplus_{g = f + \square} H_g$ given by $x \otimes y \mapsto \bigoplus \sqrt{\lambda_g} a_{gf} y$ is an isometry; it's also surjective, since $\lambda_g > 0$, so therefore is an isomorphism.