

MODULARITY OF THE CATEGORY OF REPRESENTATIONS OF A CONFORMAL NET (PART 1)

SPEAKER: ORIT DAVIDOVICH
TYPIST: OWEN GWILLIAM

ABSTRACT. Notes from the “Conformal Field Theory and Operator Algebras workshop,” August 2010, Oregon.

Recall. A *conformal net* is a cosheaf of von Neumann algebras on S^1 that satisfies properties expected of the algebra of local observables which is covariant with respect to the $PSU(1, 1)$ -action.¹

Example. The vacuum conformal net associated to a loop group.

A representation of a conformal net is a compatible family $\{\pi_I : I \subset S^1\}$.

Example. Every positive energy representation of the vacuum conformal net associated to a loop group.

We will now make extra assumptions on our conformal nets:

- (1) separability;
- (2) split property;
- (3) strong additivity.

For loop groups, (1) is obvious, and Wassermann proves (2) and (3) in his paper.

There are two equivalent forms of the split property:

- if $\bar{I} \cap \bar{J} = \emptyset$, then $\mathcal{A}(I) \otimes \mathcal{A}(J) \xrightarrow{\cong} \mathcal{A}(I) \vee \mathcal{A}(J) \subset B(H_0)$
- if $\bar{I} \subset J$, then there exists a type I von Neumann algebra N such that $\mathcal{A}(I) \subset N \subset \mathcal{A}(J)$.

Date: August 20, 2010.

Available online at <http://math.mit.edu/~eep/CFTworkshop>. Please email eep@math.mit.edu with corrections and improvements!

¹Don't take this definition too seriously. It may not literally be a cosheaf, just a precosheaf, and we don't allow ourselves to evaluate on the whole circle.

We want to consider separable representations (i.e., the Hilbert space is separable) and nondegenerate representations (for any interval I and $x \in \mathcal{A}(I)$, if $\pi(x)\zeta = 0 \Rightarrow \zeta = 0$).

Remark: The identity in $\mathcal{A}(I)$ may not go to the identity in a representation π . All we know is that it goes to a projection that commutes with $\pi(\mathcal{A}(I))$.

Claim. π is nondegenerate if and only if $\pi_I(1) = \mathbf{1}_{\mathcal{H}}$.

Let $Rep_S(\mathcal{A})$ denote the category of nondegenerate separable representations of \mathcal{A} .

Claim. If the μ_2 -index is finite, then $Rep_S(\mathcal{A})$ is a modular tensor category!

Goal: Show $Rep_S(\mathcal{A})$ is semisimple. That is it has finitely many irreducible representations and every representation is completely reducible.

We'll assume the finite index condition from hereon.

Definition. A representation π of \mathcal{A} is *localized* at $I \subset S^1$ if

- π is defined on \mathcal{H}_0 (always the vacuum Hilbert space – the defining representation of \mathcal{A})
- $\pi_{I'} = \mathbf{1}$

Proposition 0.1. For $\pi \in Rep_S(\mathcal{A})$, π is unitarily equivalent to a representation localized in any interval $I \subset S^1$.

Proof. We'll need the following facts about Type III_1 factors:

- (1) they are simple as algebras;
- (2) every representation on a separable Hilbert space is also continuous with respect to the strong operator topology;
- (3) hence $\mathcal{A}(I) \hookrightarrow \pi_I(\mathcal{A}(I))$, and
- (4) $\pi_1(\mathcal{A}(I))$ is a von Neumann algebra so $\mathcal{A}(I) \cong \pi_I(\mathcal{A}(I))$.

Yoh told us that any two representations of a type III factor are unitarily equivalent. Thus there exists a unitary map $u \in Hom(\mathcal{H}_0, \mathcal{H})$ such that $u\pi_{I'_0}(x) = xu$ for all $x \in \mathcal{A}(I'_0)$.

Define $\rho_I(x) = u\pi_I(x)u^*$ for all $x \in \mathcal{A}(I)$. By construction ρ is localized at I_0 and is equivalent to π . \square

Note. ρ localized at I_0 insures that $\rho(\mathcal{A}(I)) \subset \mathcal{A}(I)$ for every $I \supset I_0$.

Proof. Haag duality says $\mathcal{A}(I) = \mathcal{A}(I)'$. Thus $\rho_I(\mathcal{A}(I))$ commutes with $\mathcal{A}(I') = \rho_{I'}(\mathcal{A}(I'))$. Use locality in \mathcal{A} and regularity. \square

Dimension: π an irreducible in $Rep_S(\mathcal{A})$. Inclusion of type III_1 factors implies $\pi(\mathcal{A}(I)) \subset \pi(\mathcal{A}(I'))'$. The index $[\pi(\mathcal{A}(I'))' : \pi(\mathcal{A}(I))] \in [1, \infty]$.

Example. M a factor then $M \hookrightarrow Mat_k(M)$ by diagonal matrices, and $[Mat_k(M) : M] = k^2$.

Proposition 0.2. *For π an irrep, the index does not depend on the interval I .*

Heuristic idea behind proof: the index should only depend on the equivalence class, so since $\pi \cong \rho$ localized on any interval I , then the index should not depend on the interval I .

Remark. Corbett assumed that representations were conformally invariant, but if this index is finite, then the representation is conformally invariant (though we might not know this *a priori*).

Definition. For an irrep π , the *statistical dimension* $d(\pi)$ is the square root of the index.

Theorem 0.1 (Longo). *The statistical dimension and the quantum dimension agree.*

Thus the index is the square of the quantum dimension, so if you prefer the categorical notion of quantum dimension, you can stick to that.

Remark. if the index equals one, then a representation is invertible. That is, you can find another representation that tensors with it to give the unit.

Now let's consider the inclusion of two disjoint intervals I and J . Let $E = I \cup J$. Then $\mathcal{A}(E) = \mathcal{A}(I) \vee \mathcal{A}(J) \subset B(\mathcal{H}_0)$. We have inclusion of type III_1 factors $\mathcal{A}(E) \subset \mathcal{A}(E)'$.

Proposition 0.3. *The index $[\mathcal{A}(E) : \mathcal{A}(E)']$ does not depend on E .*

Definition. $\mu_2(\mathcal{A})$ is this index.

Claim. *Given the extra assumptions on our conformal net \mathcal{A} , if $\mu_2(\mathcal{A}) < \infty$, then $\mu_2(\mathcal{A})$ gives an upper bound on the number of isomorphism classes of irreducible representations in $Rep_S(\mathcal{A})$.*

Remark. $\mu_2 = \sum_i d(\pi_i)^2$ where i runs over isomorphism classes of irreps. (Note that $d(\pi_i) \geq 1$.)

Complete reducibility:

We want to make the universal enveloping C^* algebra of our cosheaf. There is a subalgebra $C_S^*(\mathcal{A})$ related to separability.

If π is a separable nondegenerate rep of $C^*(\mathcal{A})$ on \mathcal{H} , then we can decompose $\mathcal{H} = \int_X^\oplus \mathcal{H}_x d\nu(x)$ and $\pi = \int_X^\oplus \pi_x d\nu(x)$.

Let $R \subset \pi(C^*(\mathcal{A}))''$ act diagonally on \mathcal{H} . There are two important possibilities: R is the center, and R is a maximal abelian

Here's the strategy to disintegrate the representation:

- (1) use the central decomposition
- (2) show all the π_x are type I
- (3) the π_x are multiples of an irreducible
- (4) outside the null set, $x \neq x'$ then π_x and $\pi_{x'}$ are not multiples of the same irrep
- (5) there are only finitely many irreps so $|X|$ is finite and thus our representation decomposes into a sum