BOUNDARY CFTS AND THEIR CLASSIFICATION VIA FROBENIUS ALGEBRAS

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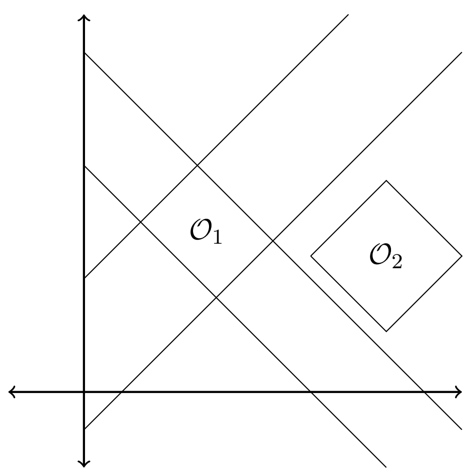
Abstract. Notes from the "Conformal Field Theory and Operator Algebras workshop," August 2010, Oregon.

Idea/Motivation: We're really doing this because we'd like to do all this in 4-dimensions, but its so intractable that we do it in 1 or 2 dimensions. Boundary CFTs are a nice intermediate point.

We always assume locality in 1-dimension: if $I \cap J = \emptyset$ then $[\mathcal{A}(I), \mathcal{A}(J)] = 0$ (they commute). In 2-dimensions, we want them to be in their causal complements.

Let $\mathcal{O}_1, \mathcal{O}_2 \in M^2$, $\mathcal{O}_1$ in the causal complement of $\mathcal{O}_2$.



[[[**Finish picture**]]] <==

Definition. Complete Rationality (of a net on $\mathbb{R}$) means you have

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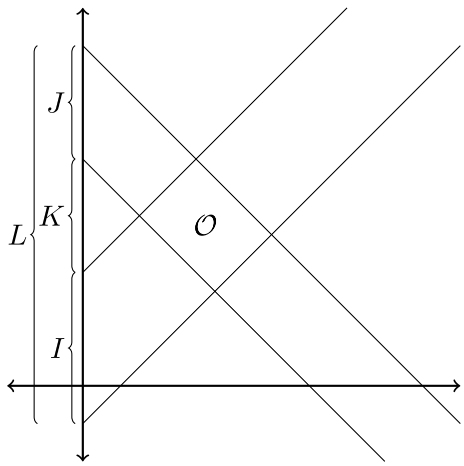
Available online at `http://math.mit.edu/CFTworkshop`. Please email `eep@math.mit.edu` with corrections and improvements!

(1) Split property: $I \cap J = \emptyset \Rightarrow \mathcal{A}(I) \vee \mathcal{A}(J) = \mathcal{A}(I) \otimes \mathcal{A}(J)$
(2) Strong additivity: $\mathcal{A}((a,b)) \vee \mathcal{A}((b,c)) = \mathcal{A}((a,c))$
(3) Finite index: $\mu_2 < \infty$

Let M_+ = positive Minkowski space = $\{(t,x) \mid x > 0\}$. Consider double cones $\mathcal{O}$



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0.1. **Boundary CFTs.**

Definition. A boundary CFT over a given 1-d conformal net $\mathcal{A}$ is an assignment $\mathcal{O} \mapsto \mathcal{B}_+(\mathcal{O}) \subset B(H)$ satisfying locality, isotony, equivariance with respect to an action of $PSL_2(\mathbb{R})$ on M_+ and H, and

- existence and uniqueness of a vacuum vector $\Omega \in B(H)$,
- covariance: $U(g)\mathcal{B}_+(\mathcal{O})U(g)^* = \mathcal{B}_+(g\mathcal{O})$ when $g\mathcal{O}$ is a double cone,
- an action π of the net $\mathcal{A}$ on H, covariant under $PSL_2(\mathbb{R})$; i.e.

$$U(g)\pi(\mathcal{A}(I))U(g^*) = \mathcal{A}(gI),$$

- Joint irreduciblity: Where $\pi(\mathcal{A})''$ = the vNA generated by all of $\pi(\mathcal{A}(I))$, then

$$\pi(\mathcal{A})'' \vee \mathcal{B}(\mathcal{O}) = B(H).$$

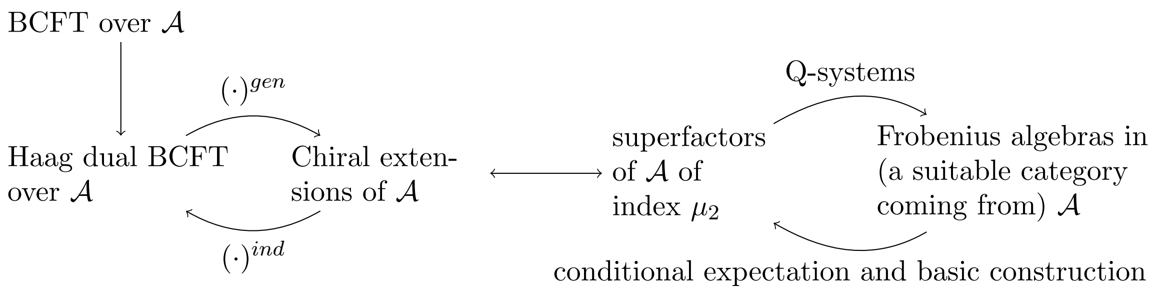
Example. Trivial BCFT over $\mathcal{A}$ is

$$\mathcal{O} \mapsto \mathcal{A}_+(\mathcal{O}) := \mathcal{A}(I) \vee \mathcal{A}(J)$$

Dual to trivial

$$\mathcal{O} \mapsto \mathcal{A}_+^{dual}(\mathcal{O}) := \mathcal{A}(K)' \cap \mathcal{A}(L).$$

0.2. **Relations between different mathematical objects.** Fix a particular 1d CN $\mathcal{A}(I)$ and examine BCFT over $\mathcal{A}$. Focus on Haag dual BCFTs over $\mathcal{A}$. These are in 1-1 correspondence with chiral extensions of $\mathcal{A}$, i.e. 1d CN which extend $\mathcal{A}$. These in turn are classified by superfactors of $\mathcal{A}$ (of index μ_2). These are in bijection with Frobenius algebras (in some category coming from $\mathcal{A}$).



The construction going from a Haag-dual BCFT over $\mathcal{A}$ to a chiral extension is *gen*, and going the other way is *incl*.

Definition. By locality $\mathcal{B}_+(\mathcal{O}) \subset \mathcal{B}_+(\mathcal{O}')'$. The BCFT $\mathcal{B}$ is *Haag dual* if this inclusion is an equality.

Definition. Given a BCFT $\mathcal{B}_+$ over $\mathcal{A}$, its boundary net $\mathcal{B}^{gen}$ is given by

$$\mathcal{B}^{gen}(I) := \mathcal{B}_+(W_I),$$

where W_I is the finite wedge determined by I, and $\mathcal{B}_+(W_I)$ is the algebra generated by $\mathcal{B}_+(O)$ for all $O \subset W_I$. This is possibly non-local, though relatively local with respect to $\mathcal{A}$; i.e.

$$[\mathcal{A}(I), \mathcal{B}(J)] = 0.$$

Theorem 0.1 (or definition)**.** *Given an irreducible (non-local) chiral extension $\mathcal{B}$ of $\mathcal{A}$, the induced BCFT*

$$B_+^{ind}(\mathcal{O}) := \mathcal{B}(L) \cap \mathcal{B}(K)'.$$

Then, we check that

- $(\mathcal{B}_+^{ind})^{gen} = \mathcal{B}$
- $(\mathcal{B}^{gen})_+^{ind} = \mathcal{B}^{dual}$

Facts:

- Given chiral extensions $I \mapsto \mathcal{B}(I) \supset \mathcal{A}(I)$, we have a consistent family of conditional expectations

$$\varepsilon_I : \mathcal{B}(I) \to \mathcal{A}(I)$$

such that $I \subset J \Rightarrow \varepsilon_I|_J = \varepsilon_J$.

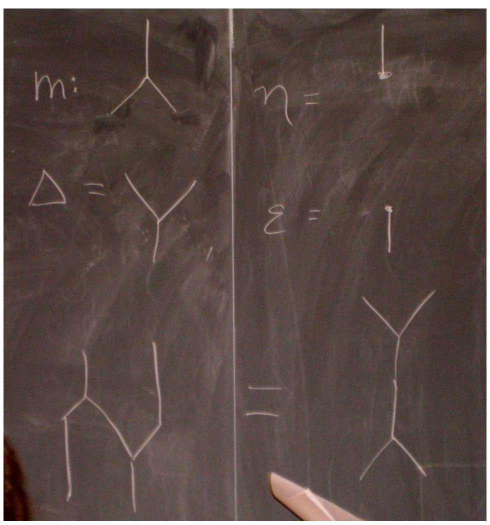
- If irreducible and finite index, then ε_I is implemented by $\varepsilon : \mathcal{B} \to \mathcal{A}$.

Theorem 0.2 (Reeh–Schlieder)**.** *The vaccum vector Ω is cyclic and separating for any $\mathcal{B}(I)$.*

Theorem 0.3. *Classifying chiral extensions $\mathcal{B}$ of $\mathcal{A}$ is equivalent to classifying "extensions" of $\mathcal{A}(I)$.*

0.3. **Superfactors to Frobenius algebras.** What is the "some category coming from $\mathcal{A}$?" Objects are elements in $\text{End}(A)$, and morphisms are intertwiners $a \in \mathcal{A}$ such that for $a \in (\rho, \sigma)$, then $a\rho(x) = \sigma(x)a$.

Frobenius algebra in a category $\mathcal{C}$ consists of an object Q, multiplication $m : Q \otimes Q \to Q$, $\eta : 1 \to Q$ such that (Q, m, η) is a monoid. $\Delta : Q \to Q \otimes Q, \epsilon : Q \to 1$ such that (Q, Δ, ϵ) is a co-monoid. And, with $I = H$ relation: $(m \otimes 1) \circ (1 \otimes \Delta) = \Delta \circ m$.



Example. G finite group with group ring $\mathbb{C}[G] = \mathbb{C}\{g \in G\}$. Then

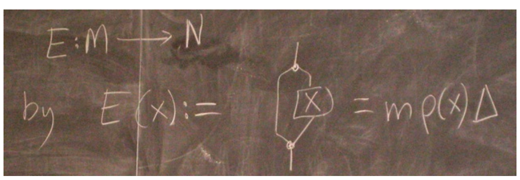
$$m(g, h) = gh$$
$$\epsilon(\sum a_g g) = a_e$$
$$\Delta(g) = \sum_{ab=g} a \otimes b$$
$$\eta(1) = 1$$

Example. Subfactors and the canonical endomorphism. Let $N \subset M$ be type III_1 subfactors, J_N, J_M the modular conjugations of N, M (with respect to a cyclic and separating vacuum vector $\Omega \in H$).

$$\begin{aligned} \gamma : M &\longrightarrow N \\ x &\longmapsto J_N J_M x J_M^* J_N \end{aligned}$$

Given $N \subset M$, we define a Frobenius algebra in $\mathrm{End}(M)$. Let γ be the canonical endomorphism $\gamma = \iota\bar{\iota}$, where $\iota : N \hookrightarrow M$, $\bar{\iota} : M \to N$.[1]

Given a Frobenius algebra in $\mathrm{End}(M)$, it gives a subfactor $E : M \to N$ by defining $E(x) = m\rho(x)\Delta$.



Proof: This is a bimodule map whose image is an algebra! Want to show

$$E(xE(y)) = E(x)E(y)$$

End proof.

[1] $\bar{\iota} = \iota^{-1} \circ \gamma$, where γ is from the example above.