OVERVIEW (MONDAY 10:15AM)

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Abstract. Notes from the "Conformal Field Theory and Operator Algebras workshop," August 2010, Oregon.

Overview: V_f, V_g are irreps of G (in this talk, we use $G = SU(N)$). We have decompositions $V_f \otimes V_g = \bigoplus_h N_{fg}^h V_h$.

Meanwhile, if H_f and H_g are irreps of $\tilde{LG}$, we have decompositions $H_f \boxtimes H_g = \bigoplus_h N_{fg}^h \cdot sign(\sigma) H_h$ (for soe σ in the affine Weyl group $\Lambda_0 \rtimes S_N$.

Note. Impliciti in this formula is semisimplicity of the Connes fusion category.

Okay, let's get down to details.

Representations of $SU(N)$:

Two principals to take for granted (cause I don't want to explain them):

1. studying complex reps of some simply connected Lie group G is the same as studying complex reps of a Lie algebra $\mathfrak{g}$.

2. complex reps of $\mathfrak{g}$ are in 1-1 correspondence with complex reps of $\mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$.

Goal: get to define the works signature, highest weight vector.

Down to business.

Example. (su =Skew hermetian 3-by-3 matrices) $su(3) \otimes \mathbb{C} = sl_3(\mathbb{C})$.

sl_3 acts on sl_3, the adjoint rep, by: Given X in first, v in second, $X(v) = [X, v] = Xv - vX$.

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please email texttteep@math.mit.edu with corrections and improvements!

This rep splits into a sum of eigenspaces; $\mathfrak{h} \subset sl_3$ is a Cartan subalgebra; $\mathfrak{h}$ = diagonal matrices in sl_3.

How does $\mathfrak{h} \circlearrowright sl_3$?

$\mathfrak{h}$ acting on $\mathfrak{h}$ kills $\mathfrak{h}$ (they commute).

Define E_{ij} for $i \neq j$ to be the matrix with one 1, in position i, j, and the rest of the entries are 0. If $X = diag(a_1, a_2, a_3)$ then $X(E_{ij}) = (a_i - a_j)E_{ij}$.

Definition. Let Ł$_i \in \mathfrak{h}^\vee = hom(\mathfrak{h}, \mathbb{C})$; $L_i : X \mapsto (X)_{ii}$. Then $X(E_{ij}) = (L_i - L_j)(X)E_{ij}$.

Picture 1: how the Cartan subalgebra acts on sl_3.

okay, now how do the off-diagonal elements of sl_3 act?

Claim. *The off-diagonal matrices, say v_β, will take some $v \in \mathfrak{g}_\alpha$ (here, $\mathfrak{g} = sl_3$ and $\alpha \in \mathfrak{h}^\vee$) and take it to $v_{\alpha+\beta} \in \mathfrak{g}_{\alpha+\beta}$. elements $\alpha \in \mathfrak{h}^\vee$ are called* weights *of the adjoint rep.*

Proof. Take $X \in \mathfrak{h}$. Need to show $X(v_\beta(v_\alpha)) = (\alpha + \beta)(X)v_\beta(v_\alpha)$. By Liebniz,

$$\begin{aligned} X(v_\beta(v_\alpha)) &= v_\beta(X(v_\alpha)) + [X, v_\beta](v_\alpha) \\ &= v_\beta(\alpha(X)v_\alpha) + \beta(X)v_\beta(v_\alpha) \\ &= (\alpha(X) + \beta(X))v_\beta(v_\alpha) \end{aligned}$$

□

More picture 1.

The upshot: $E_{ij}, i \leq j$ – the upper triangular matrices – "raise vectors". Lower triangular matrices "lower vectors". Ie, we've defined a partial order relation on these vector spaces, based on distance (and direction) from line.

Observation: there's some $\alpha \in \mathfrak{h}^\vee$ such that $\mathfrak{g}_\alpha$ is in the kernel of all raising operators.

Definition. Such an α is called the *highest weight* of a rep.

Example. $\alpha = L_1 - L_3$ is the highest weight of the adjoint representation.

Example. $sl_3 \circlearrowright \mathbb{C}^3 = V$ by $V = \mathbb{C}e_1 \oplus \mathbb{C}e_2 \oplus \mathbb{C}e_3$ and $\mathbb{C}e_1 = V_{L_1}$, $\mathbb{C}e_2 = V_{L_2}$, $\mathbb{C}e_3 = V_{L_3}$.

picture 2

Observation: The orbit of e_1 under lowering operators recovers the entire representation V. This is true in general.

Fact. V is an irrep of $sl_n(\mathbb{C})$ and v_α is the unique highest weight vector, then V is recovered by applying lowering operators of v_α.

Now, on to the idea of signature: how to find highest weight vectors.

Definition. A *signature* (called "positive weight" by people other than Wasserman) is $g \in \mathbb{Z}^N$ such that $f_1 \geq f_2 \geq \cdots \geq f_N \geq 0$.

Question. Is there a rep of $sl_N(\mathbb{C})$ such that the highest weight vector has weight $\Sigma f_i L_i$?

Answer. Yes! Take $e_f = (e_1)^{\otimes(f_1-f_2)} \otimes (e_1 \wedge e_2)^{\otimes(f_2-f_3)} \otimes \cdots (e_1 \wedge \cdots \wedge e_n)^{\otimes(f_n)} \in V^{\otimes(\Sigma f_i)}$.

Example. In the adjoint action $sl_3 \circlearrowright sl_3$, the highest weight vector is $V_\alpha, \alpha = L_1 - L_3$. What's the signature? $(1, 0, -1)$ is not a valid signature; fortunately it's equal to $(2, 1, 0)$. This is because:

Fact. $f = (a, a, \ldots, a) + f$ as the vector sum from $(a, a, \ldots, a)$ is zero.

Definition. *Young diagrams*: picture 3

Given a signature f, the associated Young diagram to f has f_i boxes in row 2.

Theorem 0.1. Pieri rule*: Notation:* $[k] = (1, 1, \ldots, 1)$ *– young diagram having k vertical boxes.* $V_f \otimes V_{[k]} = \bigoplus_{g \geq_k f} V_g$ *where* $g \geq_k f$ *is obtained by adding k boxes to f, without adding two boxes in any row.*

Example. pic 4

Question. What is trivial rep?

Remark. When looking at reps of $\tilde{LSU}(N)$, signatures need to be permissible, ie, $f_1 - f_N \leq \ell$.

Theorem 0.1. *Verlinde Formula: If*

$$V_f \otimes V_g = \bigoplus N_{fg}^h V_h$$

then

$$H + f \boxtimes H_g = \bigoplus_h N_g h^h sign(\sigma_h) H_{h'}.$$

(Go back and forth between $SU(N)$ and loops by looking at the rep where the auxilery action on the circle, acts trivially). Here h' is obtained from h but must be permissible. Details in Wasserman

Action of affine Weyl group:

Definition. $\Lambda_0 = \{(N+\ell)(m_i) | (m_i) \in \mathbb{Z}^n, \Sigma m_i = 0\}$; S_N =symm group on N letters.

The *affine Weyl group* is $\Lambda_0 \rtimes S_n$ (translations and reflections).

$$\begin{tikzcd} L_2 & & \\ & \mathfrak{h} \arrow[r] & L_1 \end{tikzcd}$$

Example.

Question.

Answer.

Definition.