OPERATOR ALGEBRAS AND CONFORMAL FIELD THEORIES WORKSHOP: DAY 1, TALK 4

Contents

1. Fermionic Fock space: Ryan Grady

We've seen reps of loop groups. Now we construct a particular rep of LU(N) via a section of the trivial bundle $S^1 \times \mathbb{C}^N \to S^1$. We'll use

- Clifford algebras
- Fock reps

Let $(H, \langle , \rangle)$ be a $\mathbb{C}$-Hilbert space, $\mathrm{Cliff}(H)$ a unital algebra with involution. We can generate $\mathrm{Cliff}(H)$ using a linear map $c : H \to H$ satisfying

$$c(f)c(g) + c(g)c(f) = 0$$

and

$$c(f)c(g)^* + c(g)^*c(f) = \langle f, g \rangle \, 1$$

We can also realize $\mathrm{Cliff}(H)$ as a quotient of the tensor algebra ($T(H) = \bigoplus_i H^{\otimes i}$),

$$\mathrm{Cliff}(H) = T(H)/(f \otimes f - \langle f, f \rangle)$$

(Note: this c is Wassermann's a.)
$\mathrm{Cliff}(H)$ has a rep on $\bigwedge H$ (wave representation),

$$\pi(c(f))(x) := f \wedge x$$

and we have a cyclic vector $\Omega = 1 \in \bigwedge^0 H = \mathbb{C}$. The c is creation, a is annihilation, $a(f) := c(f)^*$, then

$$a(f)(\omega_0 \wedge \cdots \wedge \omega_n) := \sum_{j=0}^{n} (-1)^j \langle f, \omega_j \rangle \, (\omega_0 \wedge \cdots \wedge \widehat{\omega}_j \wedge \cdots \wedge \omega_n)$$

<u>Fact:</u> $a(f), c(f)$ are adjoint wrt

$$\langle \omega_0 \wedge \cdots \wedge \omega_n, \eta_0 \wedge \cdots \wedge \eta_n \rangle = \det(\langle \omega_i, \eta_j \rangle)$$

Proposition 1.1. *$\bigwedge H$ is irreducible as a $\mathrm{Cliff}(H)$ representation.*

Proof. For $T \in \mathrm{End}(\bigwedge H)$ which commutes with all $a(f)$'s, then $T\Omega = \lambda\Omega$ (follows as $\bigcap \ker a(f) = \mathbb{C}\Omega$). If, in addition, T commutes with all $c(f)$'s, then $T = \lambda I$. □

Comment $\bigwedge H$ is a Hilbert space completion of the direct sum of the finite exterior powers of H.
So, this is a representation, how do we get more?

1.1. Unitary structure.

Let $(V, (,))$ be a $\mathbb{R}$-Hilbert space. A unitary structure is a $J \in O(V)$ such that $J^2 = -I$. V_J is a $\mathbb{C}$-Hilbert space where multiplication by i is multiplication by J and the (Hermitian) inner product is

$$\langle v, w\rangle := (v, w) + i(v, Jw)$$

Now define a projection operator

$$P_J := \frac{1}{2}(I - iJ) \in \mathrm{End}(V_J)$$

Definition. The *Fermionic Fock space*, $\mathcal{F}_P$ is

$$\mathcal{F}_P := \bigwedge(PH)\widehat{\otimes}\bigwedge(P^{\perp}H)^*$$

(where $H = V_J$) That is, take $\bigwedge(PH), \bigwedge(P^{\perp}H)$, take Hilbert space completion, then form the tensor product of the Hilbert spaces. Thiss is an irreducible representation of $\mathrm{Cliff}(V_J)$,

$$\pi_P(c(f)) = c(Pf) \otimes 1 + 1 \otimes c((P^{\perp}f)^*)^*$$

Note: H is a complex Hilbert space, defining P is equivalent to defining a new complex structure where PH is the i-eigenspace and $P^{\perp}H$ is the $-i$-eigenspace.

Theorem 1.2 (I. Segal-Shale equivalence criterion)**.** *The Fock reps π_P and π_Q are unitarily equivalent if and only if $P - Q$ is Hilbert-Schmidt (i.e. $\{e_i\}$ a basis for H, $\sum \|(P - Q)e_i\|^2 < \infty$.)*

$u \in U(H)$ induces an automorphism of $\mathrm{Cliff}(H)$, $c(f) \mapsto c(uf)$. We say it is *implemented* in π_P (or $\mathcal{F}_P$) if

$$\pi_P(c(uf)) = U\pi(c(f))U^*$$

for some unitary $U \in U(\mathcal{F}_P)$ unique up to scalar.

Proposition 1.3. *$u \in U(H)$ is implemented if and only if $[u, P]$ is Hilbert-Schmidt.*

Definition. The *restricted unitary group* is

$$U_{res} = \{u \in U(H) \mid u \text{ is implemented in } \mathcal{F}_P\}$$

Thus, we have a representation $U_{res}(H) \to PU(\mathcal{F}_P)$, the "basic representation."

If $u \in U(H)$ and $[u, P] = 0$, then u is implemented in $\mathcal{F}_P$ and is "canonically quantized."

1.2. **A representation of LU(N).** Let $H = L^2(S^1) \otimes \mathbb{C}^N$, $P : H \to H_{\geq 0}$ where $H_{\geq 0}$ is the space of functions with non-negative Fourier coefficients (boundary values of functions on the disk). For $f \in C^\infty(S^1, \mathrm{End}(\mathbb{C}^N))$, multiplication by f defines an operator $m(f)$ on H.

Fact: $\|[P, m(f)]\|_2 \leq \|f'\|_2$.
So, in particular, $f \in LU(N)$ is implemented in $\mathcal{F}_P$. Thus, we have a projective representation of $LU(N)$ on $\mathcal{F}_P$. This is the "fundamental rep." Thinking of the circle as the boundary of a conformal disk, we look at the group of Moebius transformations, i.e. the transformations of the circle which preserve the conformal structure.

$$SU_{\pm}(1,1) = \left\{ \begin{pmatrix} \alpha & \beta, \\ \overline{\beta} & \overline{\alpha} \end{pmatrix} : |\alpha|^2 - |\beta|^2 = \pm 1 \right\}$$

is a double cover of the Moebius group.
For $g \in SU_+(1,1)$,

$$g(z) = \frac{\alpha z + \beta}{\overline{\beta} z + \overline{\alpha}}$$

gives a unitary action on $H = L^2(S^1) \otimes \mathbb{C}^N$,

$$(V_g \cdot f)(z) = \frac{f(g^{-1}(z))}{\alpha - \overline{\beta} z}$$

(g is the action on the circle, V_g is the induced action on the Hilbert space.)
Note: $|z| < 1$, $|\alpha| > |\beta|$, then $(\alpha - \overline{\beta} z)^{-1}$ is holomorphic so V_g commutes with P so the action is canonically quantized.
For $F = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$,

$$(V_F \cdot f)(z) = \frac{f(z^{-1})}{z}$$

so $V_F P V_F = I - P$.

We have a $U(1)$ action on H given by multiplication by z. This commutes with projection so is canonically quantized: let U_z be the action on $\mathcal{F}_P$.

Proposition 1.4. *If π is the rep of $LSU(N)$ on $\mathcal{F}_P$ and U_z is the $U(1)$ action on $\mathcal{F}_P$, then*

$$\pi(g) U_z \pi(g)^* = U_z$$