THE CENTRAL EXTENSION OF LG, POSITIVE ENERGY REPRESENTATIONS, LIE ALGEBRA COCYCLES

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ABSTRACT. Notes from the "Conformal Field Theory and Operator Algebras workshop," August 2010, Oregon.

The talk will have three main parts:

(1) define & motivate positive energy representations;
(2) projective representations & central extensions;
(3) reducing questions to the loop algebra.

Throughout the talk, G denotes a compact connected Lie group and $\mathbb{T}$ the circle as a group.

1. POSITIVE ENERGY REPRESENTATIONS

At first acquaintance, the definition of positive energy representation probably seems a little weird and unmotivated. We'll get the awkward introduction to them out the way before explaining why we'll be spending so much time with PERs this week.

Definition. A *positive energy representation* (PER) of LG is a topological vector space E with:

(1) a projective representation $LG \to PU(E)$, meaning that for $\gamma \in LG$ you can choose $U_\gamma \in GL(E)$ (not continuously) and then $U_\gamma U_{\gamma'} = c_{\gamma\gamma'} U_{\gamma\gamma'}$, where $c_{\gamma\gamma'} \in \mathbb{C}^*$;
(2) an intertwining action of $\mathbb{T}$, so that R_θ, the operator for $\theta \in \mathbb{T}$ on E, satisfies $R_\theta U_\gamma R_\theta^{-1} = U_{R_\theta \gamma}$;

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(3) under the weight decomposition of E by $\mathbb{T}$,[1]
$$E = \bigoplus_{n\in\mathbb{Z}} E(n),$$
with $\mathbb{T}$ acting by $e^{in\theta}$ on $E(n)$, we have
$$\begin{cases} E(n) = 0, & n < 0, \\ \dim E(n) < \infty, & n \geq 0. \end{cases}$$
This is known as the *positive energy condition.*

1.1. **Concrete examples.** Ryan's talk provides the crucial example of a PER for this week and he's already verified the relevant properties. Let $G = SU(n)$, $V = \mathbb{C}^n$ the standard representation, $\mathcal{H} = L^2(S^1, V)$, and $P : \mathcal{H} \to \mathcal{H}$ projection onto non-negative Fourier modes. Then the fermionic Fock space
$$\mathcal{F}_P = \Lambda(P\mathcal{H})\hat{\otimes}\Lambda(P^{\perp}\mathcal{H})^*$$
is a positive-energy representation. In fact, this is a *level one* representation, and $(\mathcal{F}_P)^{\otimes \ell}$ is a positive-energy representation of level ℓ.

1.2. **Why study PERs?** There are two types of reasons for focusing our attention on PERs: mathematical and "physical." From the point of view of math, the most convincing reason is that PERs are well-behaved.

Theorem 1.1 (9.3.1 in [PS])**.** *A positive energy representation of LG is*

- *completely reducible into irreducible positive energy representations,*
- *unitary,*
- *and extends to holomorphic representations of its complexification $LG_{\mathbb{C}} := Maps(S^1, G_{\mathbb{C}})$.*

Furthermore, the representation admits a projective intertwining action of $\mathrm{Diff}_+(S^1)$.[2]

This theorem tells us that a PER acts a lot like a representation of a compact Lie group, since the first three properties are what make life so nice when studying compact Lie groups. The last property insures that reparametrizing the circle does not affect a PER: the PERs are preserved under a huge space of LG automorphisms. In a sense, being a PER is a kind of finiteness condition that insures we can get complete control over the representation theory.

[1] Note that this action is not the U_z action in Wassermann, but is the rotation subgroup of the Möbius group.

[2] This is the group of diffeomorphisms of the circle that preserve orientation.

Although I'm know next to nothing about physics, I'll explain why I guess that the positive energy condition is natural from a physics point of view, and I hope that this week will provide better explanations. Here are three thoughts:

- We focus on projective representations because the space of states for a quantum mechanical system is really $\mathbb{P}(H)$, the projective space of the Hilbert space H, so it's only reasonable to expect that a group act projectively on H.
- When studying quantum mechanics, the "energy" of a particle corresponds to the infinitesimal generator of translation along its worldline (this is what Schrödinger's equation says). When studying QM along a circle, the analog of translation is rotation. The positive energy condition is then the physically reasonable assertion that the accessible energy states are bounded below.
- Segal says that a PER is the "boundary condition" of a holomorphic representation of the semigroup $\mathbb{C}^\times_{\leq 1} \rtimes \widetilde{LG}_\mathbb{C}$. That is, PERs extend to the study of maps of the annulus into $G_\mathbb{C}$. Thus PERs ought to capture at least the annular part of a CFT (at least in the Segal style).

2. Projective representations and central extensions

We now explore what a projective representation is and how to construct invariants that classify such representations.

Definition. A *projective unitary representation*[3] of a group G on a Hilbert space V is a group homomorphism

$$\rho : G \to PU(V) := U(V)/\mathbb{T},$$

where PU denotes the projective unitary group obtained by quotienting out by the subgroup of scalar multiples of the identity.

We would like to lift this to an actual representation, which will require taking a central extension of the group. That is, we pull back the short exact sequence defining PU:

$$\begin{tikzcd}
1 \arrow[r] & \mathbb{T} \arrow[r] \arrow[d, equal] & U(V) \arrow[r] & PU(V) \arrow[r] & 1 \\
1 \arrow[r] & \mathbb{T} \arrow[r] & \widetilde{G} \arrow[r] \arrow[u, dashed, "\rho"] & G \arrow[r] \arrow[u, "\rho"] & 1
\end{tikzcd}$$

[3]The unitary group uses the strong operator topology (or any equivalent topology) and then PU has the quotient topology. You would get in trouble if you used the norm topology to define $U(V)$.

Thus, given a projective representation $\rho : G \to PU$, we obtain an extension $\widetilde{G}$ and an honest representation $\rho : \widetilde{G} \to U$. Conversely, given an honest representation of a central extension of G, we get a projective representation of G. Thus we can just study extensions of G and honest representations to get a handle on projective representations.

Another consequence is that we obtain an invariant for a projective representation: the associated central extension. We can give an invariant for an extension in two ways.

Idea 1: Thinking like topologists, we note that any extension

$$1 \to \mathbb{T} \to \widetilde{LG} \to LG \to 1$$

defines a circle bundle and hence an element

$$c_1\Big(\widetilde{LG} \to LG\Big) \in H^2(LG;\mathbb{Z}).$$

We can try to use this cohomological information to help construct group extensions or narrow down possible ones. (There's a really great story if you head down this road of mixing topology and representation theory, leading to twisted K-theory, Chern-Simons, and so on, but we'll not pursue it.)

Homework. Check that $H^2(LSU(N);\mathbb{Z}) \cong \mathbb{Z}$. How does the level of $\mathcal{F}_P$ compare to the first Chern class of the associated extension?

Idea 2: Thinking like algebraists, we note that at the Lie algebra level we get a central extension

$$0 \to \mathbb{R} \to \widetilde{L\mathfrak{g}} \to LG \to 0.$$

The extended Lie bracket has the form

$$\Big[(X,a),(Y,b)\Big] = \Big([X,Y],\omega(X,Y)\Big)$$

for $(X,a),(Y,b) \in L\mathfrak{g} \oplus \mathbb{R} \cong \widetilde{L\mathfrak{g}}$ (as vector spaces). Consequently, the functional $\omega : L\mathfrak{g} \times L\mathfrak{g} \to \mathbb{R}$ must be skew-symmetric and satisfy a Jacobi identity, i.e.

$$\omega([X,Y],Z) + \omega([Y,Z],X) + \omega([Z,X],Y) = 0.$$

This implies ω will be a 2-cocycle and live in $H^2(L\mathfrak{g},\mathbb{R})$.

There is a beautiful characterization of such cocycles.

Proposition 2.1 (4.2.4 of [PS])**.** *For $\mathfrak{g}$ semisimple, every continuous G-invariant 2-cocycle ω for $L\mathfrak{g}$ has the form*

$$\omega(X,Y) = \frac{1}{2\pi}\int_0^{2\pi} \langle X(\theta), Y'(\theta)\rangle \, d\theta$$

where $\langle\cdot,\cdot\rangle$ is a $\mathfrak{g}$-invariant symmetric bilinear form on $\mathfrak{g}$.

The proof is quite simple: you simply observe that such a cocycle extends to the complexification $L\mathfrak{g}_{\mathbb{C}}$ and apply Fourier series. It's important to observe that the G-invariant condition is irrelevant. Since LG acts on $L\mathfrak{g}$ by the adjoint action, it acts on cocycles, and two cocycles are cohomologous if they live on the same orbit. Now notice that the constant loops $G \subset LG$ act on cocycles as well, but since G is compact, we can do the averaging trick to replace any cocycle by a cohomologous G-invariant cocycle.

Thanks to this theorem, it's enough to know about symmetric $\mathfrak{g}$-invariant bilinear forms on the finite-dimensional Lie algebra $\mathfrak{g}$. This is a well-known (to others, probably!) piece of mathematics. In particular, $H^3(\mathfrak{g}, \mathbb{R})$ is isomorphic to the space of symmetric bilinear $\mathfrak{g}$-invariant forms on $\mathfrak{g}$, via the map

$$\begin{aligned} \mathrm{Sym}^2(\mathfrak{g})^G &\longrightarrow H^3(\mathfrak{g}) \\ \langle \cdot, \cdot \rangle &\longmapsto \langle \cdot, [\cdot, \cdot] \rangle. \end{aligned}$$

(There's something interesting here in relation to the Chern-Simons action, but I'm not competent enough to comment ...)

It's natural to ask now whether given these invariants (either topological or Lie-theoretic), we can reconstruct a central extension of LG. For instance, every 2-cocycle on $L\mathfrak{g}$ yields a 2-form on LG by left translation. This would be the first Chern class of the extension if it existed, so we know this 2-form has to be integral. If you follow this train of thought and have familiarity with the Chern-Weil story of bundles, curvature, and connections, then the following theorem is unsurprising.

Theorem 2.1 (4.4.1 of [PS])**.** *Let G be compact and simply-connected. Then*

- *A 2-cocycle ω on $L\mathfrak{g}$ defines a group extension if and only if $[\omega/2\pi] \in H^2(LG; \mathbb{Z})$.*
- *If it defines a group extension, the group extension is unique.*
- *If G is simple, such an ω is an integral multiple of ω_{basic}, where ω_{basic} is a rescaling of the Killing form so that the longest coroot has length 2.*

Example. For $G = SU(N)$, the Killing form is ω_{basic}:

$$\langle X, Y \rangle = -\operatorname{Tr}(XY).$$

Note that for $SU(2)$, there is only one coroot and we verify

$$\langle \theta, \theta \rangle = -\operatorname{Tr}\left(\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}^2 \right) = 2.$$

Definition. If G is simple, the *level* of a central extension $\widetilde{LG}$ of LG is the integer $\ell \in \mathbb{Z}$ such that $\omega_{\widetilde{LG}} = \ell \omega_{\text{basic}}$. For simply-connected semisimple G, the level ℓ lives in $H^3(G; \mathbb{Z}) \cong H^4(BG; \mathbb{Z})$.

2.1. **Redefining PERs.** With the language of central extensions now available to us, we can give a more succinct definition of a positive energy representation. We denote any central extension of LG by $\widetilde{LG}$, and if we want to specify the level ℓ extension, we use $\widetilde{LG}_\ell$. Thus in the definition of PER, we replace "projective representation of LG" by "representation of $\widetilde{LG}$." Likewise, saying "there is an intertwining action of $\mathbb{T}$" means that the rotation action of $\mathbb{T}$ on LG plays nicely with the representation. This property can be rephrased by saying the PER is a representation of the semidirect product

$$1 \to \widetilde{LG} \to \widetilde{LG} \rtimes \mathbb{T} \to \mathbb{T} \to 1,$$

where $\mathbb{T}$ acts by rotation on $\widetilde{LG}$. In sum, we obtain the following.

Definition. A *positive energy representation* of LG at level ℓ is an honest representation of $\widetilde{LG}_\ell \rtimes \mathbb{T}$ satisfying the positive-energy condition for the action of $\mathbb{T}$ by rotation.

Once you pick a maximal torus for G (and a splitting of the semidirect product), you have the abelian subgroups

$$\mathbb{T}_{\text{rot}} \times T_G \times \mathbb{T}_{CE} \subset \widetilde{LG} \rtimes \mathbb{T}_{\text{rot}}.$$

There are many circle actions appearing this week, so I've tried to distinguish the circle from the central extension, $\mathbb{T}_{CE}$, from the rotation circle, T_{rot}. They play different roles so watch out!

A positive energy representation decomposes under the action of this big torus into a finer sum of weight spaces

$$E = \bigoplus_{(\text{energy } n, \lambda, \text{ level } l)} E(n, \lambda, \ell),$$

where λ is the weight for the action of T_G.

Notice that for an irreducible PER, only one level can appear by Schur's lemma: the action of $\mathbb{T}_{CE}$ is central, and hence commutes with everything in sight, so the decomposition of E by level is preserved under the full action of $\widetilde{LG}$. In consequence, when thinking about irreducible PERs, we can restrict our attention to the weight space for $T_G \times \mathbb{T}_{\text{rot}}$, and this is quite useful in understanding the affine Weyl group.

3. Reducing to the loop algebra

The loop group is a big, rather unwieldy thing, and we'd like a smaller, more algebraic object to work with. A first step, familiar from Lie theory, is to work with its Lie algebra, but even $L\mathfrak{g}$ is too unwieldy. Instead, we introduce

$L^{\text{poly}}\mathfrak{g}$, the polynomial loops into $\mathfrak{g}$, for which explicit computations are easy. This is the space of smooth maps into $\mathfrak{g}$ that are given by trigonometric polynomials:

$$L^{\text{poly}}\mathfrak{g} := \left\{ \sum_{\text{finite}} X_n \cos(n\theta) + Y_n \sin(n\theta) \ : \ X_n, Y_n \in \mathfrak{g} \right\}$$

Clearly it is dense in $L\mathfrak{g}$. Its complexification is simply

$$L^{poly}\mathfrak{g} \otimes \mathbb{C} \cong \mathfrak{g}_{\mathbb{C}}[z, z^{-1}],$$

where $z \leftrightarrow e^{in\theta}$.

We want a correspondence between representations of LG and representations of $L^{\text{poly}}\mathfrak{g}$. That is, we want a bijection:

$$\{\text{PERs of } LG\} \longleftrightarrow \{\text{PERs of } L^{\text{poly}}\mathfrak{g}\}$$

It is not so hard to construct a map going to the right, but there is some work involved in showing it's a bijection. Once we have the bijection, we can often reduce questions about LG to purely algebraic computations with $L^{\text{poly}}\mathfrak{g}$ (for instance, see Nick's talk on the action of the Virasoro).

Abstractly, a PER E of LG induces a representation of $L^{\text{poly}}\mathfrak{g}$ in the obvious way:

$$L^{\text{poly}}\mathfrak{g} \xrightarrow{\exp} LG \xrightarrow{\pi} PU(E).$$

For each $x \in L^{\text{poly}}\mathfrak{g}$ you get a 1-parameter subgroup

$$\begin{aligned} \mathbb{R} &\longrightarrow PU(E) \\ t &\mapsto \pi(\exp(tx)). \end{aligned}$$

By Stone's theorem, we get a skew-adjoint operator $X \in \text{End}(E)$ such that

$$e^{tX} = \pi(e^{tx}).$$

(This depends on picking a lift to $U(E)$, so X is defined up to a character of $\mathbb{R}$.) Thus we have a projective representation

$$\rho : L^{\text{poly}}\mathfrak{g} \to \operatorname{End} E.$$

Wassermann showed this representation is well-behaved.

Let d denote the infinitesimal generator of $\mathbb{T}_{\text{rot}}$ on E, so that

$$d\Big|_{E(n)} = n \quad \& \quad R_\theta = e^{i\theta d}.$$

Theorem 3.1 ([Was]). *Let E be a level ℓ representation of LG. Then*

- *The space of* finite energy vectors $E^{\text{fin}} \subset E$ *is preserved by ρ;*[4]

[4] This is the dense subspace of E given by taking the algebraic direct sum of the weight space $E^{\text{fin}} = \bigoplus_{n\geq 0}^{alg} E(n)$.

- *We can choose lifts (of the 1-parameter subgroups) such that*
$$[d, \rho(x)] = i\rho(x')$$
on E^{fin}*;*
- $[\rho(x), \rho(y)] = \rho([x, y]) + i\ell\omega_{basic}(x, y)$.

This theorem says that we get a level ℓ PER for $L^{\text{poly}}\mathfrak{g}$. Notice that the second property says the Lie algebra of rotations intertwines correctly with our PER for $L^{\text{poly}}\mathfrak{g}$, and the third property says the level is respected.

In the paper [Was], Wassermann proves this theorem in a very concrete way. The proofs usually boil down to a simple combination of functional analysis and explicit computation. Recall our concrete example of PERs at the beginning of the talk.

Let $\mathcal{H} = L^2(S^1, V) = L^2(S^1, \mathbb{C}^N)$, and recall that $\mathrm{Cliff}(\mathcal{H})$ acts on the fermionic Fock space $\mathcal{F}_P$.[5] We thus obtain an action of polynomial loops in V on $\mathcal{F}_P$:
$$\begin{aligned} L^{\text{poly}}V = \{\text{polynomials in } e^{in\theta}\} \otimes V &\xrightarrow{\pi} \mathrm{Cliff}(\mathcal{H}), \\ e^{in\theta} \otimes v &\longmapsto a(e^{in\theta}v). \end{aligned}$$
We want a map
$$L^{\text{poly}}su(N) \xrightarrow{\pi} \mathrm{Cliff}(\mathcal{H})$$
which would give us a PER for $L^{\text{poly}}su(N)$.

Observe that $su(N)_{\mathbb{C}} = sl(N, \mathbb{C})$ is generated by the elementary matrices E_{ij}. It is enough to say where π would send these generators. Let $\{e_1, \ldots, e_n\}$ be the standard basis for $V = \mathbb{C}^N$. Set $e_k(n) = \pi(e^{-in\theta} \otimes e_k)$, and
$$E_{ij}(n) = \sum_{m>0} e_i(n-m)e_j(-m)^* - \sum_{m\geq 0} e_j(m)^* e_i(m+n).$$
Theorem 3.2 ([Was])**.**

- $[X(m), a(f)] = a(Xe^{im\theta}f)$ *for* $f \in L^{\text{poly}}V$*.*
- $[d, X(m)] = -mX(m)$
- $[X(n), Y(m)] = [X, Y](n+m) + m\langle x, Y\rangle \delta_{n+m,0}$
- $\langle Xe^{-in\theta}, Ye^{-in\theta}\rangle = \int \langle X, Y\rangle_{\text{basic}} e^{-in\theta}(-im)e^{-im\theta}\, d\theta$*, and therefore you have a level 1 representation.*

The proof consists primarily of straightforward computation. You can then extend it to $\mathcal{F}_p^{\otimes \ell}$ to obtain the level ℓ representations of $LSU(N)$.

[5]The action uses Wassermann's notation. It would be $c(e^{in\theta}v)$ in Ryan's notation.

References

[PS] Andrew Pressley and Graeme Segal. *Loop groups*. Oxford Mathematical Monographs. The Clarendon Press Oxford University Press, New York, 1986.

[Was] Antony Wassermann. Operator algebras and conformal field theory. III. Fusion of positive energy representations of LSU(N) using bounded operators. *Invent. Math.*, 133(3):467–538, 1998.