OVERVIEW (MONDAY 9AM)

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Abstract. Notes from the "Conformal Field Theory and Operator Algebras workshop," August 2010, Oregon.

Question. What is conformal field theory?

Many possible answers: **operator algebras**, vertex algebras, Graham-Segal formalism.

Conformal field theory is a vague notion you might interpret concretely to mean "conformal net"

Question. is there some way to pass between the various versions?

Answer. yes; constructions that work in some cases, with extra assumptions. Some of these are conjectured but not known to work. There are also results that are known in some settings and only conjectured in others.

Many known examples. Most interesting ones come from loop groups; also Dirac-Fermion ("free fermion" in Wasserman) examples.

Definition. Let's describe these loop group nets. G is a simply connected compact Lie group ($SU(n)$ in Wasserman's paper). $LG = \mathrm{Map}_{C^\infty}(S^1, G)$. $\tilde{LG}$ is a central extension of LG by S^1; this depends on a choice of $\ell \in \mathbb{Z}$, called the "level." We also have an action $\tilde{LG}$ on $\mathcal{H}_0$; in order for this to have the right properties, we need $\ell > 0$.

Where do the von Neumann algebras come in? Given $I \subset S^1$, consider $L_I G = \{\gamma : S^1 \to G | \gamma(z) = e \forall z \notin I\}$.

Define $\tilde{L_I} G$ to be the pullback of the central extension $\tilde{LG} \to LG$ along the inclusion $L_I G \hookrightarrow LG$:

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$$\begin{tikzcd}
\tilde{L_I G} \arrow[r, hook] \arrow[d, two heads] & \tilde{LG} \arrow[d, two heads] \\
L_I G \arrow[r, hook] & LG
\end{tikzcd}$$

$\tilde{L_I G}$ acts on $\mathcal{H}_0$; so set $\mathcal{A}(I) :=$ von Neumann algebra generated by $\tilde{L_I G}$.

So: $I \mapsto \mathcal{A}(I)$ is a conformal net.

Axioms include: $SL_2(\mathbb{R})$ also acts on $\mathcal{H}_0$. There's a particular element $\Omega \in \mathcal{H}_0$, the vacuum vector, which is the unique vector such that $g\Omega = \Omega \forall g \in SL_2(\mathbb{R})$. (why $SL_2(\mathbb{R})$? It is symmetries of circle).

Summary: For all G, ℓ, we have a conformal net.

Wasserman's paper: it's had to tell which things are true for $SU(N)$ and which hold in general. But, $SU(N)$ turns out to be a good example; things that are true there

He then goes on to study representation of the conformal net, without naming them as such. Representation of the conformal net are representations of the loop group, provided we have positive energy.

When $I \subset J$, $\mathcal{A}(I) \hookrightarrow \mathcal{A}(J)$; a representation of a conformal net is a choice of another hilbert space, $\mathcal{H}_\lambda$ together with a homomorphism $\mathcal{A}(I) \mapsto \mathcal{B}(\mathcal{H}_\lambda)$ so that

$$\begin{tikzcd}
\mathcal{A}(I) \arrow[rr] \arrow[dr] & & \mathcal{B}(H) \\
& \mathcal{A}(J) \arrow[ur] &
\end{tikzcd}$$

commutes.

The difference between $\mathcal{H}_o$ and $\mathcal{H}_\lambda$ is that $\mathcal{H}_\lambda$ doesn't have a vacuum vector (and, $SL_2(\mathbb{R})$ doesn't exactly act on $\mathcal{H}_\lambda$; need an extra central extension).

Now, the local loop groups $\tilde{L_I G}$ act on $\mathcal{H}_\lambda$; so $\tilde{LG}$ is also represented, ie LG has a projective representation.

Characterize these representations:

Reps of the loop group net are in 1-1 correspondence with projective reps of LG which are positive energy. (something about level having to do with the projective reps.)

What is the big idea in Wasserman's paper? Defining a tensor product operator on these loop group representations, using techniques from von Neumann algebras.

Definition. When one works with von Neumann algebras, one has the following construction: ${}_AH_B$ is a vNa bimodule (H a hilb. space, A, B vNa's with commuting actions.) There is a construction in vNa which behaves formally like tensor product: Connes fusion, ${}_AH \boxtimes_B K_C$.

Warning! It's not true that given $h \in H$ and $k \in K$, one obtains an element of $H \boxtimes K$.

Definition. Locality: For intervals I_1 and I_2 with $I_1 \cap I_2 = \emptyset$, $\mathcal{A}(I_1)$ and $\mathcal{A}(I_2)$ commute.

So, do I and its complement I' make $\mathcal{H}_\lambda$ a bimodule? Not quite: both act on left, even though they commute. But, if $I \simeq J$ is orientation-reversing, $\mathcal{A}(I) \simeq \mathcal{A}(J)^{op}$. $\mathcal{A}(I) \simeq \mathcal{A}(I')^{op}$; so we have ${}_{\mathcal{A}(I)}(\mathcal{H}_\lambda)_{\mathcal{A}(I)}$.

Thus: two representations of a conformal net can be tensored to produce a new one!

Question. Are intervals open?

Answer. Doesn't matter if they're open or closed. What's not allowed is to consider a single point, or all of S^1.

Okay, but what do we get? Given irreducible H_μ, H_λ, $H_\lambda \boxtimes H_\mu \simeq \bigoplus_\nu N^\nu_{\lambda,\mu} H_\nu$. What are the N? This is the hard work that Wasserman does.

The answer is quite pretty:

Theorem 0.1. *$(Rep(LG_\ell), \boxtimes)$ forms a ring under fusion; this ring is a quotient of $(Rep(G), \otimes)$ with kernel explicitly defined.*

This seems to be true for general groups, too. But, in conformal nets, this in not a general theorem (has been done for other examples, eg $SO(n)$).

Question. Is this the same as the representation category of the quantum group at a root of unity?

Answer. Of course! But this in not written down anywhere. I don't know whether anyone knows how to do it. Later this week, we'll hear about how this is a modular category.

Question. Is the dependence on ℓ only in the kernel?

Answer. Yes, because ...