THE SYMMETRIC MONOIDAL 3-CATEGORY OF CONFORMAL NETS

SPEAKER: BRAXTON COLLIER
TYPIST: BRENT PYM

Abstract. Notes from the "Conformal Field Theory and Operator Algebras workshop," August 2010, Oregon.

The goal of this talk is to give a definition of symmetric monoidal 3-categories, show that conformal nets form such a category and relate the subject to Chern-Simons theory. (This is work with Bartles, Douglas and Henriques.)

1. Motivation: Topological QFT

Let $Bord^n_{n-1}$ be the category of bordisms of n-manifolds. This category is a symmetric monoidal category with operation given by disjoint unions and identity object the empty set $\varnothing$.

Definition. An n-dimensional topological quantum field theory or TQFT is a symmetric monoidal functor

$$Z : Bord^n_{n-1} \to (Hilbert, \otimes)$$

$\Omega Bord^n_{n-1} = End(\varnothing) =$ closed n-manifolds/diffeomorphisms

$\Omega Hilb = End(\mathbb{C}) = \mathbb{C}$

So Z assigns $\mathbb{C}$-valued diffeomorphism invariant to closed n-manifolds.

$Bord^n_k$ is the symmetric $(n-k)$-category which we think of as manifolds with bordisms of bordisms of ..., We have

$$\Omega Bord^n_k \cong Bord^n_{k+1}$$

Definition. Let $\mathcal{C}$ be a symmetric monoidal n-category with $\Omega^{n-1}\mathcal{C} \cong Hilb_{\mathbb{C}}$. A *$\mathcal{C}$-valued local TQFT* is a symmetric monoidal functor $Bord^n_0 \to \mathcal{C}$

Date: August 24, 2010.

Available online at `http://math.mit.edu/∼eep/CFTworkshop`. Please email `eep@math.mit.edu` with corrections and improvements!

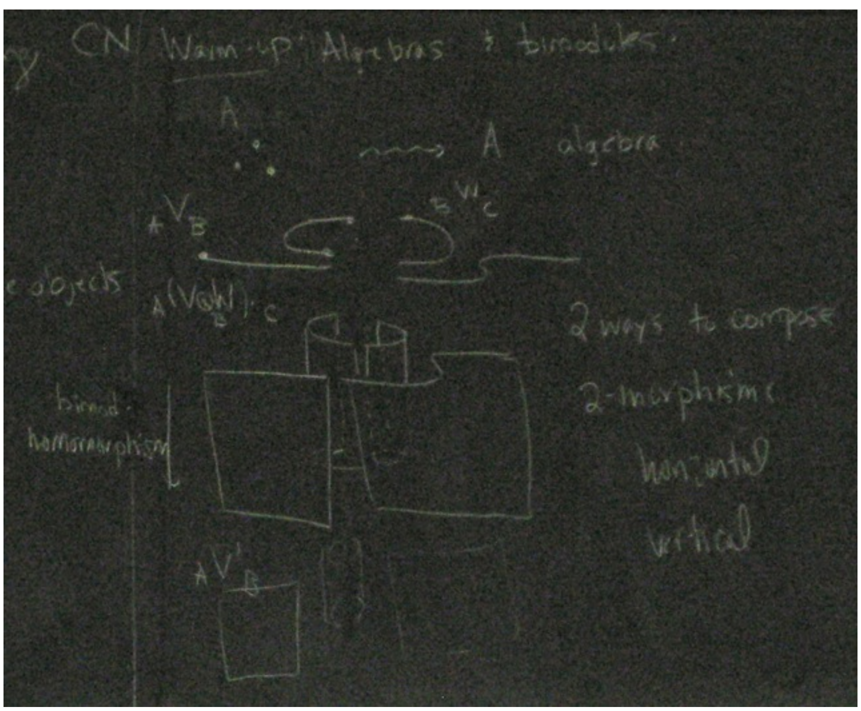
Theorem 1.1 (BDH)**.** *There exists a symmetric monoidal 3-category CN whose objects are conformal nets and $\Omega^2 CN \cong Hilb$.*

Theorem 1.2 (Cobordism hypotheis of Baez-Dolan, 99% certainty proof by Hopkins-Lurie)**.** *Framed local $\mathcal{C}$-valued n-dimensional TQFTs are in one-to-one correspondence with dualizable objects in $\mathcal{C}$.*

Theorem 1.3. *$A \in CN$ is dualizable if and only if it is the direct sum of irreducible conformal nets with finite μ-index.*

2. Warm-up: algebras and bimodules

We assign to a 0-dimensional manifold an algebra A. Given a 1-morphism between points with algebras A and B, we assign an (A, B)-bimodule V. We compose these bimodules using the tensor product. To a 2-morphism, we assign a bi-module homomorphism between the corresponding bi-modules. There are two ways to compose these morphisms (horizontal and vertical) and in this case we ask that they agree.



Remark. We can thing of a symmetric monoidal category as a bi-category with a 1-object, so a symmetric monoidal category is something at least 4-categorical in nature.

There is a symmetric monoidal category of algebras with $\otimes_{\mathbb{C}}$. There is also a symmetric monoidal category of bimodules with $\otimes$:

$$({}_AV_B) \otimes \left({}_{A'}V'_{B'}\right) = {}_{A\otimes A'}(V \otimes V')_{B\otimes B'}.$$

In this case the functors s and t with take an arrow to its source and target are symmetric monoidal functors $Bimod \to Alg$ with

$$\boxtimes : Bimod\ {}^s\!\times^t\ Bimod \to Bimod$$

The upshot is that $(Alg, Bimod)$ is a category object in the 2-category SMC of symmetric monoidal categories. Let us, then, think of conformal nets as a bicategory object in SMC.

3. Conformal nets revisited

Definition. We call Int the (topological) category whose objects are oriented intervals and whose morphisms are smooth embeddings, which are not necessarily orientation preserving. The topology on the hom-sets is given by point-wise convergence.

Notice that we are allowing more information than just the transformation given by Mobiüs transformations.

Definition. A conformal net is a continuous functor

$$\mathcal{A} : Int \to vN\text{-}alg$$

from intervals to von Neumann-algebras satisfying the usual axioms, as well as

- For $\phi : I \to I$, such that ϕ is the identity in a neighbourhood of ∂I, then $\mathcal{A}(\phi) : \mathcal{A}(I) \to \mathcal{A}(I)$ is inner
- If $\phi : I \to J$ is orientation preserving (resp. reversing) then $\mathcal{A}(\phi)$ is a homomorphism (resp. anti-homomorphism).

4. Conformal nets and 2-algebras

Given a conformal net $\mathcal{A}$, let $A = \mathcal{A}([0,1])$. Define the standard inclusions

$$i, j : [0,1] \to [0,2]$$

where i is the inclusion and j is inclusion plus translation one unit to the right. Also pick an isomorphism

$$s : [0,2] \to [0,1]$$

which has derivative 1 in a neighbourhood of $\partial[0,2]$. We define

$$\mu : A \times A \to A$$

by

$$\mu(x,y) = s_*(i_*(x)j_*(y)) = (si)_*(x)(sj)_*(y) = \begin{pmatrix} x \\ y \end{pmatrix}$$

Claim. *There exists $v \in A$ such that*

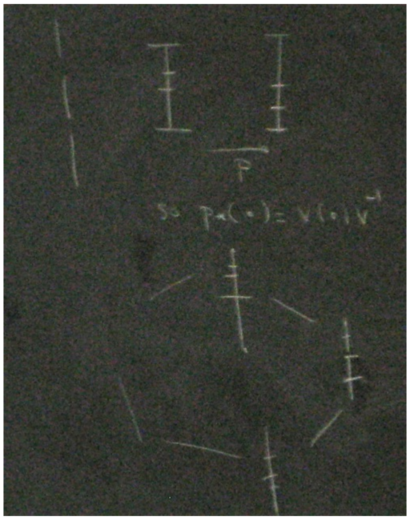
(1)

$$v \left(\begin{pmatrix} x \\ y \end{pmatrix} \\ z \right) v^{-1} = \left(x \\ \begin{pmatrix} y \\ z \end{pmatrix} \right)$$

(2)

$$v^2 = \begin{pmatrix} 1 \\ v \end{pmatrix} v \begin{pmatrix} v \\ 1 \end{pmatrix}$$

We get identities like



The morphisms in the category are:

- 1-morphisms: defects
- 2-morphisms: sectors
- 3-morphisms: homomorphisms of sectors

Definition. A *bicolored interval* is an interval I with two subintervals I_w and I_b (the *white* and *black* intervals) with $I = I_w \cup I_b$ and such that either

(1) $I_w = \varnothing$,
(2) $I_b = \varnothing$, or
(3) $I = I_w \cup I_b$

together with a coordinate function $c : nbd(I_w \cap I_b) \to \mathbb{R}$.

We can define a category Int_{bc} of bicolored intervals.

Definition. A *defect* $D : A \to B$ *for* $A, B \in CN$ is a cosheaf $D : Int_{bc} \to vN\text{-}alg$ such that

$$D|_{\text{white int.}} = A$$

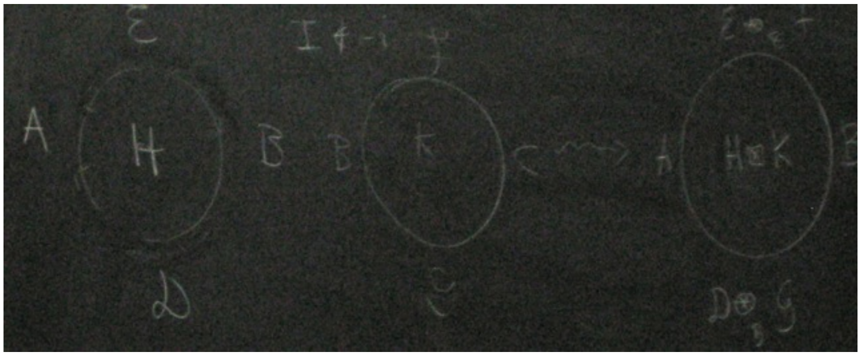
and

$$D|_{\text{black int.}} = B$$

5. Sectors

Consider intervals I in S^1 such that either $i \notin I$ or $-i \notin I$. We can bicolor such intervals: call the pieces to the left of $\pm i$ black and those to the right of $\pm i$ white.

We have bimodules with defects and a composition using Connes fusion:



There is a natural isomorphism between the different ways of fusing, which uses the machinery we've been discussing this week.