PRIMARY FIELDS, BOUNDEDNESS OF SMEARED PRIMARY FIELDS

SPEAKER: ARTURO PRAT-WALDRON
TYPIST:

Abstract. Notes from the "Conformal Field Theory and Operator Algebras workshop," August 2010, Oregon.

Goal: $H_\lambda, H_\mu \in$ IPER; we want to study their Connes fusion, relatedly their intertwiners. These are an honest rep of $LG \tilde{\rtimes} \mathbb{T}_{rot}$. $I \subset S^1$.

We want to study $\mathrm{Hom}^{bndd}_{L_I G}(H_\lambda, H_m u)$ and get explicit elements.

Example. $G = SU(N)$, $V = \mathbb{C}^N$ the vector representation. $H = L^2(S^1, V)$ and $Cliff(H) \circlearrowleft \mathcal{F}_V$

$LG \rtimes \mathbb{T}_{rot} \to U_{res}(H) \to PU(\mathcal{F}_V)$

We have $H_\lambda, H_\mu \subset \mathcal{F}_V^{\otimes \ell}$ and P_λ, P_μ projections onto these subspaces.

Given $f \in L^2(S^1, V)$, send it to creation $a(f) \in B(\mathcal{F}_V^{\otimes \ell})$.

we have an equivariance condition, $\pi(g)a(f)\pi(g)^* = a(gf)$, with $g \in LG \tilde{\rtimes} \mathbb{T}_{rot}$ and $\pi : LG \tilde{\rtimes} \mathbb{T}_{rot} \to U(\mathcal{F}_V^{\otimes \ell})$.

Define $\phi_{\lambda,\mu}(f) = P_\mu a(f) P_\lambda^* \in \mathrm{Hom}^{bdd}(H_\lambda, H_\mu)$.

If $g \in \tilde{L_I G}$ and $supp(f) \subset I^c$ then $gf = g$; So $\pi(g)\phi(f)\pi(g)^* = \phi(f)$. This implies $\phi(f) \in \mathrm{Hom}_{\tilde{L_I G}}(H_\lambda, H_\mu)$; we also have boundedness, $\|\phi(f)\| \leq \|f\|_{L^2}$.

This gives us a procedure for construction bndd intertiwners; start with a function, construct an operator; as long as it's equivariant, we get an element of the intertwining space.

Date: August 19, 2010.
Available online at `http://math.mit.edu/~eep/CFTworkshop`. Please email `eep@math.mit.edu` with corrections and improvements!

Definition. fields take functions on S^1 and produce an element of the intertwiner $\mathrm{Hom}(H_\lambda, H_\mu)$. *Primary* means they are $LG \rtimes \mathbb{T}_{rot}$ equivariant.

Definition. V is a G-module, H_λ, H_μ are IPERs of level ℓ. Define $V^{fin} := V[z, z^{-1}]$ which is acted on by $L^{poly}\mathfrak{g}_\mathbb{C} \rtimes i\mathbb{R}d$. A *primary field of charge V and level* ℓ is a linear map

$$\phi : V[z, z^{-1}] \otimes H_\lambda^{fin} \to H_\mu^{fin}$$

which is $L^{pol}\mathfrak{g}_\mathbb{C} \rtimes i\mathbb{R}d$ equivariant.

$H_\lambda^{fin} = \bigoplus_{N\geq 0}^{alg} H_\lambda(n)$ are modes of the primary field ϕ. $\phi(v,n) := \phi(v \otimes^n) : H_\lambda^{fin} \to H_\mu^{fin}$. The $L^{pol}\mathfrak{g}_\mathbb{C} \rtimes i\mathbb{R}d$ equivariance is equivalent to

(1) $[X[n], \phi(v,m)] = \phi(Xv, m+n)$ (2) $[d, \phi(v,n)] = -n\phi(v,n)$

For $X \in \mathfrak{g}_\mathbb{C}$, $X(n) = X \otimes z^n \in L^{pol}\mathfrak{g}_\mathbb{C}$.

(2) implies $\phi(v,n)$ lowers energy by n: $\phi(v,n) : H_\lambda(k) \to H_\mu(k-n)$. (check: $d(\phi(v,n)\xi) = \phi(v,n)d\xi - n\phi(v,n)\xi = (k-n)\phi(v,n)\xi$.)

In particular, $\phi(v,0) : H_\lambda(0) \to H_\mu(0)$.

$\phi_0 : V \otimes V_\lambda \to V_\mu$ in the initial term of ϕ.

Equation (1) implies equivariance w.r.t. $\mathfrak{g}$, which impies $\phi_0 \in \mathrm{Hom}_G(V \otimes V_\lambda, V_\mu)$.

Lemma 0.1. *The map*

$$\mathrm{Hom}_{L^{pol}\mathfrak{g}_\mathbb{C} \rtimes i\mathbb{R}d}(V^{fin} \otimes H_\lambda^{fin}, H_\mu^{fin}) \to \mathrm{Hom}_G(V \otimes V_\lambda, V_\mu)$$

coming from $\phi \mapsto \phi_0$ *is injective.*

Proof. H_λ^{fin} is generated by the $H_\lambda(0)$ operators as a $L^{pol}(\mathfrak{g}_\mathbb{C})$-module. □

Denote the image of $\mathrm{Hom}_{L^{pol}\mathfrak{g}_\mathbb{C} \rtimes i\mathbb{R}d}(V^{fin} \otimes H_\lambda^{fin}, H_\mu^{fin})$ in $\mathrm{Hom}_G(V \otimes V_\lambda, V_\mu)$ by $\mathrm{Hom}_G^\ell(V \otimes V_\lambda, V_\mu)$.

Proposition 0.2. $\mathrm{Hom}_G^\ell(V \otimes V_\lambda, V_\mu) = \mathrm{Hom}_G(V_\lambda \otimes V_\mu, V_\nu)$ *if* λ, μ, ν *are admissible at level* ℓ *and at least one is* minimal *G-module: means highest weight is dominant. (audience doesn't think that's what this means. Miniscule? something about exterior algebra?)*

1. $G = SU(N)$. Vector primary fields of $SU(N)$:

Definition. vector=charge of primary field is the vector rep $V = V_\square = \mathbb{C}^N$.

If f, g are signatures of admissible G-modules of level ℓ, we want to study $\mathrm{Hom}_{SU(N)}(V \otimes V_f, V_g)$.

Consider $V_\square \otimes V_f = \bigoplus_{g>f} V_g$ where g can be obtained from f by adding one box.

Let $W = V \otimes \mathbb{C}^\ell$; V injects into W. $\Lambda W = (\Lambda V)^{\otimes \ell}$ and

$$\begin{aligned} S : W \otimes (\Lambda W) &\to \Lambda W \\ w \otimes x &\mapsto w \wedge x \end{aligned}$$

$V_f, V_g \subset (\Lambda V)^\ell$.

Lemma 1.1. *Let $T \in \mathrm{Hom}_{SU(N)}(\square \otimes V_f, V_g)$ and $T \neq 0$. WE can find $SU(N)$-equivariant projections $P_\square : W \to V_\square$, $P_f : W \to V_f$, $P_g : W \to V_g$, such that $T = P_g S(P_\square^* \otimes P_f^*)$.*

Proof. consider a signature $f = (f_1 \geq f_2 \geq \cdots \geq f_N)$ which is admissible, ie $f_1 - f_N \leq \ell$. Let $(e_1, \ldots e_N)$ be a basis of $V_\square$.

We let $e_f = e_1^{\otimes(f_1 - f_2)} \otimes (e_1 \wedge e_2)^{\otimes(f_2 - f_3)} \cdots (e_1 \wedge \cdots \wedge e_N)^{\otimes(\ell - f_1 + f_N)}$.

for e_g we have a similar formula; the only way we can get a non-zero intertwiner is when g has one more box than f. ie, $g_i = f_i$ if $i \neq k$ and $g_k = f_k + 1$. We get from e_f to e_g by adding $\pm e_k$ in the $(f_1 - f_k)$ copy of $\Lambda V \subset (\Lambda V)^{\otimes \ell}$.

Example. (On camera)

Now take $P_\square : W = V_\square \otimes \mathbb{C}^\ell = \bigoplus_\ell V_\square \to V_\square$, projection onto the $(f_1 - f_k)$ copy of $V_\square$.

We have $SU(N)e_f \hookrightarrow V_f \subset \Lambda W$, similarly for G, and P_f or P_g going the other direction: $\Lambda W \to V_f$.

$S(P_\square^* \otimes) : V_\square \otimes (\Lambda W) \to \Lambda W$ (recall $\Lambda(W) = (\Lambda V)^{\otimes \ell}$) which implies esterior multiplication by an element of $V_\square$ in the $f_1 - f_k$ copy of $\Lambda V \subset (\Lambda V)^{\otimes \ell}$.

□

Theorem 1.2. *Any $SU(N)$-intertwiner $\phi_0 : V_\Box \otimes H_\lambda(0) \to H_\mu(0)$ is the initial term of a unique vector primary field. All vector primary fields arise as "compressions of fermions" so satisfy the L^2 bound $\|\phi(f)\| \leq c\|f\|_2$ for all $f \in V[z, z^{-1}]$. the map extends continuously to $L^2(S^1, V_\Box)$ and satisfies the global equivariance relation $\phi_\mu(G)\phi(f)\pi_\lambda(g)^* = \phi(gf)$ for all $g \in (LG \tilde{\rtimes} \mathbb{T}_{rot})_\ell$*

Proof. $\Lambda W = \mathcal{F}_W(0)$, $W \subset L^2(S^1, W)$ as constant functions – $V_\lambda, V_\mu \subset W$, $H_\lambda, H_\mu \subset \mathcal{F}_W$ giving projections P_λ, P_μ.

Define $\phi_{\lambda,\mu}(v, n) := P_\mu a(v \otimes z^n)P_\lambda^*$. □

Punchline: if I have a smeared primary field, I evaluate it on a complementary interval to get one of these intertwiners.