CONNES FUSION

SPEAKER: YOH TANIMOTO
TYPIST: EMILY PETERS

Abstract. Notes from the "Conformal Field Theory and Operator Algebras workshop," August 2010, Oregon.

The plan is to relate Connes fusion and endomorphisms.

In this talk, M is always a type III factor. For our purposes, it suffices to have the following property:

Fact. Any representation of M on a separable Hilbert space, is implemented by a unitary operator. Another way of saying this is that any two representations are equivalent.

Definition. An (M, M) bimodule is a Hilbert space X with commuting actions of M and M^{op}.

Definition. An endomorphism of M is a unital *-homomorphism of M into M.

Example. $L^2(M)$ is a trivial bimodule. For $x, y \in M$ and $\xi \in L^2(M)$, $x \cdot \xi \cdot y$.

Example. If ρ is an endomorphism of M, then it also acts on $L^2(M)$. We define $\rho(x) \cdot \xi \cdot y = \rho(x) J Y^* J \xi$. Call this bimodule X_ρ.

Proposition 0.1. *Any bimodule is unitarily equivalent to some* X_ρ.

Proof. From the first fact, as representation of M^{op}, X and $L^2(M)$ are equivalent. We may assume that $X = L^2(M)$ as M^{op}-modules. The action of M commutes with M^{op}. $(M^{op})' = M$; the image of M is M. $\square$

Proposition 0.2. $X_{\rho_1} \simeq X_{\rho_2}$ *iff there is a* $u \in \mathcal{U}(M)$ *such that* $u\rho_1(x)u^* = \rho_2(x)$.

Date: August 19, 2010.

Available online at `http://math.mit.edu/~eep/CFTworkshop`. Please email `eep@math.mit.edu` with corrections and improvements!

Proof. in one direction, u commutes with M^{op}. In the other, let u implement the equivalence. u must commute with $M^{op} = M'$, so $u \in M$. □

	direct sum	subobject	
bimodule	$X \oplus Y$	invariant subspace	fusion
endomorphisms	$P_1 \perp P_2$, $P_1 + P_2 = I$, $v_I : P_i \simeq I$: $V_1\rho_1(x)V_1^* + V_2\rho_2(x)V_2^*$	$P \in M$, $[P, \rho(x)] = 0$, $V : P \simeq I$. $V\rho(x)V^*$.	composition $\rho_2 \circ \rho_1$.

Let X, Y be bimodules. $\mathcal{X} = \mathrm{Hom}(L^2(M)_M, X_M)$ and $\mathcal{Y} = \mathrm{Hom}({}_M L^2(M), {}_M Y)$.

We consider $\mathcal{X} \otimes \mathcal{Y}$ with an inner product, $\langle x_1 \otimes y_1, x_2 \otimes y_2 \rangle = \langle x_2^* x_1 y_2^* y_1 \Omega, \Omega \rangle$ Here we use $x_2^* x_1 \in M$ and $y_2^* y_1 \in M^{op}$. (this is because $x \in \mathrm{Hom}(L^2(M)_M, X_M)$ and $x^* \in \mathrm{Hom}(X_M, L^2(M)_M)$ gives us $x^*x \in \mathrm{Hom}(L^2(M)_M, L^2(M)_M)$ ie $x^*x \in M$.)

Lemma 0.3. *The form thus defined on $\mathcal{X} \otimes \mathcal{Y}$ is an inner product*

Proof. Show positive definiteness.

Let $z = \sum_i x_i \otimes y_i$; then $\langle z, z \rangle = \sum_{i,j} \langle x_i^* x_j y_i^* y_j \Omega, \Omega \rangle$

Now $x = (x_i^* x_j) \in M_n(M)$; rewrite it as

$$x = \begin{pmatrix} x_1^* \\ x_2^* \\ \vdots \\ x_n^* \end{pmatrix} \cdot \begin{pmatrix} x_1 & x_2 & \dots & x_n \end{pmatrix}$$

We can write $x = a^*a$ where $a \in M_n(M)$. Similarly for y, $y = b^*b$.

So, $\langle z, z \rangle = \sum_{i,j} \langle x_i^* x_j y_i^* y_j \Omega, \Omega \rangle = \sum_{i,j} \sum_{p,q} \langle a_{pi}^* a_{pj} b_{qi}^* b_{qj} \Omega, \Omega \rangle$

Now by orthogonality, all of these commute and so $\sum_{i,j} \sum_{p,q} \langle a_{pi}^* a_{pj} b_{qi}^* b_{qj} \Omega, \Omega \rangle = \sum_{p,q} \sum_{i,j} \langle a_{pj} b_{qj} \Omega, a_{qi} b_{qi} \Omega \rangle = \sum_{p,q} \| \sum_j a_{pj} b_{qj} \Omega \|^2 \geq 0$ □

We define on $\mathcal{X} \otimes \mathcal{Y}$ actions of M, M^{op} by $a, b \in M$ by $a \cdot x \otimes y \cdot b = ax \otimes Jb^*Jy$

Proposition 0.4. *These actions are well-defined.*

Definition. call the completion of $\mathcal{X} \otimes \mathcal{Y}$ the *fusion* of X and Y, $X \boxtimes Y$.

Theorem 0.5. *Let ρ_1, ρ_2 be endomorphisms of M. Then $X_{\rho_1} \boxtimes X_{\rho_2} \simeq X_{\rho_2 \circ \rho_1}$.*

Proof. The operator
$$V : x \otimes y \mapsto \rho_2(x) y\Omega$$
is a unitary. Remains to show that it's an intertwiner:
$$\begin{aligned}
&V \cdot a \cdot x \otimes y \cdot b \\
&= V\rho_1(a)x \otimes Jb^*Jy \\
&= \rho_2(\rho_1(a)x)JB^*Jy\Omega \\
&= \rho_2\rho_1(a)\rho_2(x)JB^*Jy\Omega \\
&= \rho_2\rho_1(a)Jb^*J\rho_2(x)y\Omega \\
&= \rho_2\rho_1(x)Jb^*JVx \otimes y
\end{aligned}$$

□

Corollary 0.6. $X_{\rho_1} \boxtimes (X_{\rho_2} \boxtimes X_{\rho_3}) \simeq X_{\rho_3\rho_2\rho_1} \simeq (X_{\rho_1} \boxtimes X_{\rho_2}) \boxtimes X_{\rho_3}$