QUANTUM DIMENSION

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Abstract. Notes from the "Conformal Field Theory and Operator Algebras workshop," August 2010, Oregon.

Outline for the talk

(1) An example
(2) Semi-simple ribbon categories
(3) Continuation of example
(4) Definition of quantum dimension
(5) Properties thereof
(6) Computations for IPERs of $LSU(2)$

1. Basic example

Let $V = \mathbb{C}^2$ and choose a basis $\{e_2, e_2\}$ with dual basis $\{\epsilon_1, \epsilon_2\}$ for V^*. We have an evaluation map

$$e : V \otimes V^* \to \mathbb{C}$$

defined by eating a vector with a linear functional. There is also an embedding

$$i : \mathbb{C} \to V \otimes V^*$$

defined by

$$i(1) = e_1 \otimes \epsilon_2 + e_2 \otimes \epsilon_2.$$

The composition

$$e \circ i : \mathbb{C} \to V \otimes V^* \to \mathbb{C}$$

yields

$$e(e_1 \otimes \epsilon_1 + e_1 \otimes \epsilon_2) = 2.$$

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2. Semi-simple ribbon categories

Let $\mathcal{C}$ be a $\mathbb{C}$-linear abelian category equipped with the following structure:

- $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, a bilinear functor
- $\alpha_{U,V,W} : (U \otimes V) \otimes W \xrightarrow{\sim} U \otimes (V \otimes W)$, a functorial isomorphism for all $U, V, W \in \mathcal{C}$
- $I \in \mathcal{C}$ such that $End(I) = \mathbb{C}$, together with functorial isomorphisms

  $$\lambda_V : I \otimes V \xrightarrow{\sim} V$$

  and

  $$\rho_V : V \otimes I \to V$$

  for all $V \in \mathcal{C}$

Such a structure is called a *monoidal structure*.

Example. $Vec(\mathbb{C})$, together with the tensor product $\otimes$. □

Example. $Rep(SU(N))$ with tensor product $\mathcal{H}_1 \otimes \mathcal{H}_2$ of the underlying Hilbert spaces $\mathcal{H}_1, \mathcal{H}_2 \in Rep(SU(N))$ and the action

$$g(x \otimes y) = g(x) \otimes g(y)$$

for $g \in SU(N)$ and $x \in \mathcal{H}_1$, $y \in \mathcal{H}_2$. □.

Definition. A *braiding* is given by isomorphisms

$$\sigma_{VW} : V \otimes W \xrightarrow{\sim} W \otimes V,$$

which are functorial in $V, W \in \mathcal{C}$, and satisfy some commutative diagrams. A category with a monoidal structure and a braiding is called a *braided monoidal category.*

Example. For the categories $Vec(\mathbb{C})$ and $Rep(SU(N))$, the isomorphisms

$$\tau_{VW} : V \times W \to W \times V$$

given by

$$\tau(v, w) = (w, v)$$

give a braiding. □

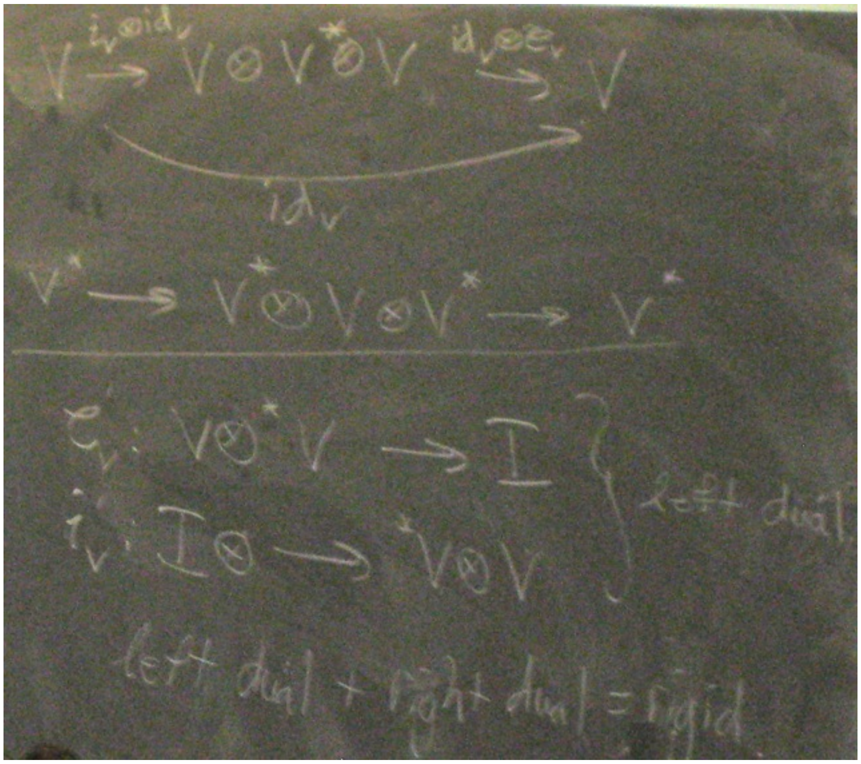
Rigidity is given by homomorphisms

$$e_V : V^* \otimes V \to I$$

and

$$i_V : I \to V \otimes V^*,$$

where V is the right dual of V. These maps must make the following diagrams commute:



Example. For $Vec(\mathbb{C})$, we have $V^* = Hom(V, \mathbb{C})$ and the maps e_V and i_V as before. □

Example. For $Rep(G)$, the representation on the dual of $\mathcal{H}$ is given by continuous linear functionals $\mathcal{H}^* = C(\mathcal{H}, \mathbb{C})$ with the representation

$$(gf)(v) = f(g^{-1}v)$$

for $f \in \mathcal{H}^*$, $g \in G$ and $v \in \mathcal{H}$. □

Suppose that we have a map $f : U \to V$. We can construct a dual map $f*$ by the composition

$$V^* \xrightarrow{\mathbf{1}\otimes i_U} V^* \otimes U \otimes U^* \xrightarrow{\mathbf{1}\otimes f\otimes \mathbf{1}} V^* \otimes V \otimes U^* \xrightarrow{e_V\otimes \mathbf{1}} U^*$$

Definition. A *ribbon category* is a rigid braided tensor category with a functorial isomorphism $\delta_V : V \xrightarrow{\sim} V^{**}$ such that

$$\delta_{V\otimes W} = \delta_V \otimes \delta_W$$
$$\delta_I = \mathbf{1}_I$$
$$\delta_{V^*} = (\delta_V^*)^{-1}$$

3. Quantum dimension

Let $\mathcal{C}$ be a semi-simple ribbon category.

Definition. Let $V \in \mathcal{C}$ and $f \in End(V)$. Define the *trace of* f to be the composition

$$I \to V \otimes V^* \xrightarrow{f \otimes \mathbf{1}} V \otimes V^* \to V^{**} \otimes V^* \xrightarrow{e_{V^*}} I$$

One important property of the trace is

$$tr(f \otimes g) = tr(f)tr(g),$$

a fact which needs the ribbon structure.

Definition. The *quantum dimension of* $V \in \mathcal{C}$ is defined to by $tr(\mathbf{1}_V)$.

If there are finitely many simple objects, the quantum dimension is a real number. For $LSU(N)$, the quantum dimension is, in fact, positive.

4. Calculations

Let

$$N_{fg}^h = \dim\left(Hom(V_h, V_f \otimes V_g)\right)$$

Theorem 4.1 (Wassermann)**.** *The Connes fusion satisfies*

$$\mathcal{H}_f \boxtimes \mathcal{H}_g = \bigoplus N_{fg}^h \mathrm{sgn}(\sigma_h) \mathcal{H}_{h'}$$

where

$$h' = \sigma(h + \delta) - \delta$$

is a permutation.

Corollary 4.2. *For $G = SU(2)$, we have*

$$\mathcal{H}_l \boxtimes \mathcal{H}_l = \mathcal{H}_0$$

Theorem 4.3. *For the permutation signature f, we have*

$$\mathcal{H}_\square \boxtimes \mathcal{H}_f = \bigoplus_{g = f + \square} \mathcal{H}_g$$

Corollary 4.4. *For $G = SU(2)$ and $1 \leq \lambda \leq l - 1$,*

$$\mathcal{H} \boxtimes \mathcal{H}_\lambda = \mathcal{H}_{\lambda - 1} \oplus \mathcal{H}_{\lambda + 1}$$

5. IPERs of $LSU(2)$

Fix a level $l \in \mathbb{Z}$, consider

$$\mathcal{H}_0, \ldots, \mathcal{H}_l,$$

and define

$$d_i = \dim \mathcal{H}_i$$

and

$$d_l = 1.$$

Proposition 5.1. *For $0 \leq i \leq l$, we have*

$$d_i = d_{l-i},$$

so that

$$d_1 d_i = d_{i-1} + d_{i+1}$$

and

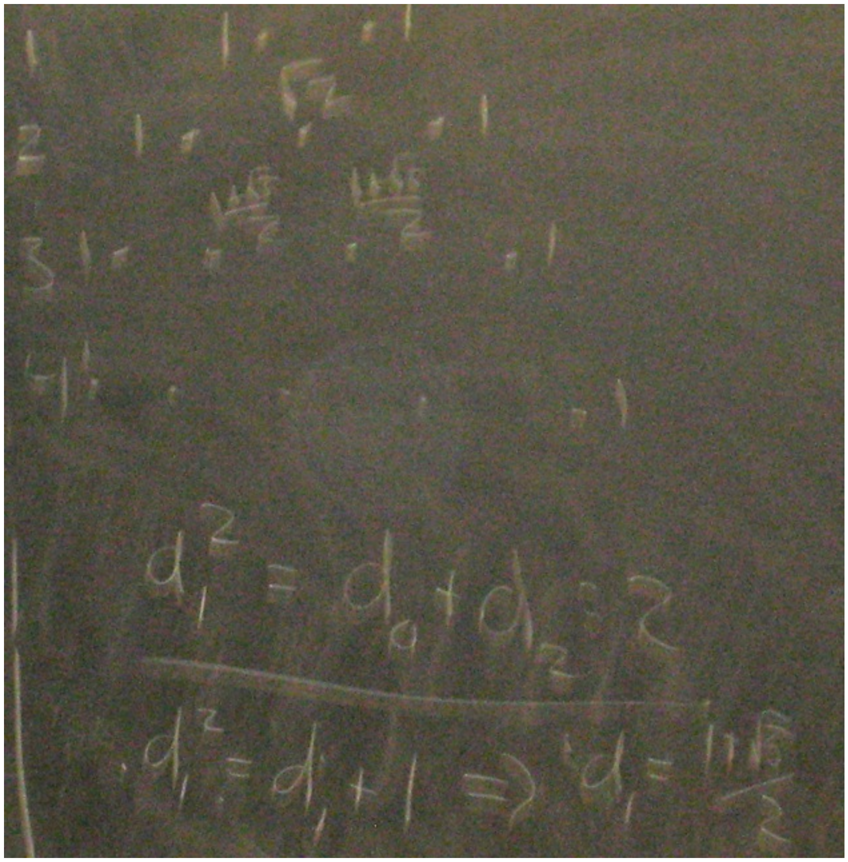
$$d_1 d_{l-i} = d_{l-i-1} + d_{l-i+1},$$

and hence

$$d_{i-1} = d_{l-i+1}$$

for $1 \leq i \leq l-1$.

We can use this proposition to calculate the quantum dimensions, as in the following picture:


$d_1^2 = d_0 + d_2 = 2$
$d_1^2 = d_1 + 1 \Rightarrow d_1 = \frac{1+\sqrt{5}}{2}$