CLASSIFICATION OF LG-REPRESENTATIONS AND WEIGHT POLYTOPES FOR THEIR CHARACTERS

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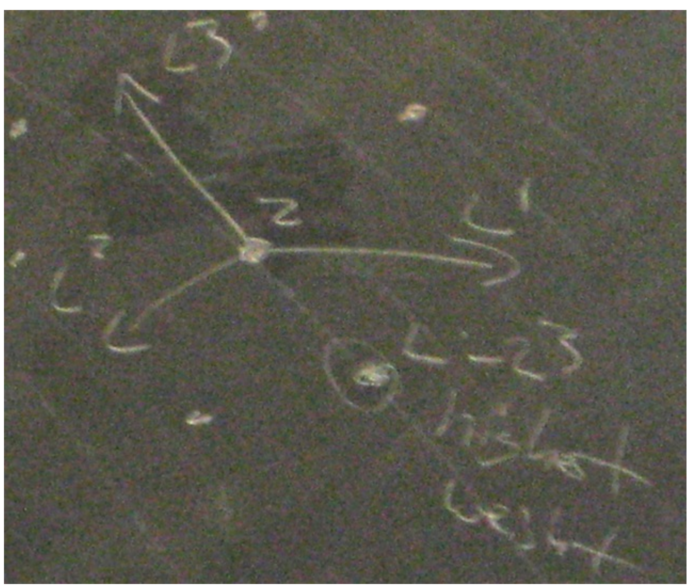
Abstract. Notes from the “Conformal Field Theory and Operator Algebras workshop,” August 2010, Oregon.

The goal is to establish a classification of the irreducible positive-energy representations (PERs). It will mirror the classification of irreducible representations of $SU(N)$. For the purposes of this talk, $N = 3$.

Recall that for $G = SU(3)$, we accept on faith that we may classify representations of G by classifying representations of $\mathfrak{su}(3)$. They were described by pictures such as



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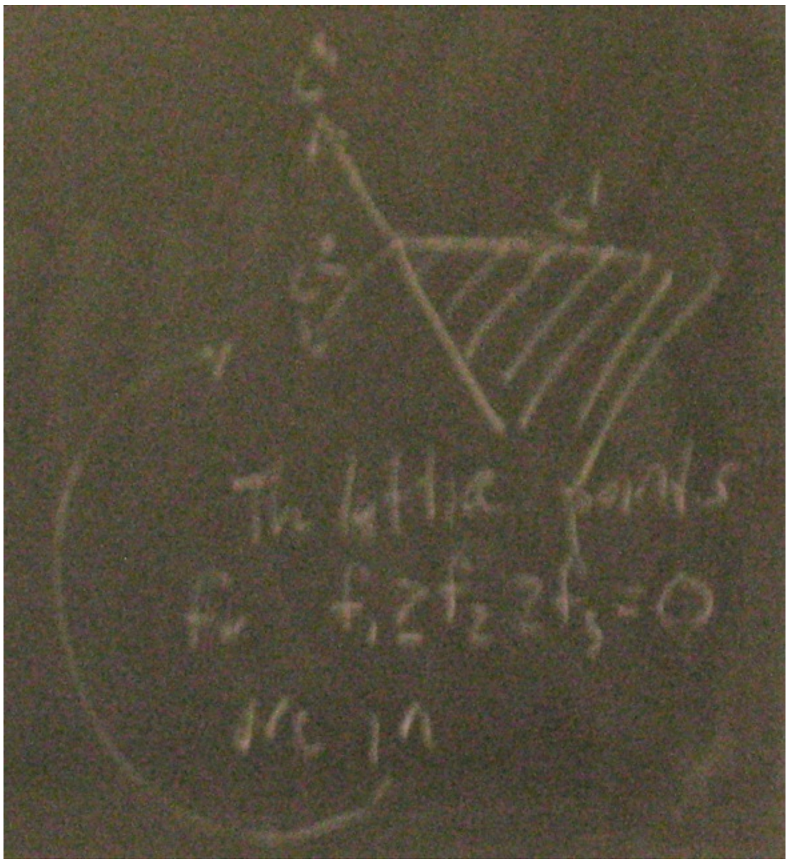
Available online at `http://math.mit.edu/∼eep/CFTworkshop`. Please email `eep@math.mit.edu` with corrections and improvements!

We think of the highest weight as a *signature* $(f_1, f_2, f_3) \in \mathbb{Z}^3$. For the adjoint representation, the signature is $(1, 0, -1)$. Recall the

Theorem. *The possible highest weights are tuples* $(f_1, \ldots, f_N) \in \mathbb{Z}^N$ *with* $f_1 \geq \cdots \geq f_N \geq 0$. *By adding an element of the form* $(a, \ldots, a)$ *we may take* $f_N = 0$.

These signatures lie in the indicated portion of the following diagram:



For $LSU(3)$, we ask the following questions:

- What is the equivalent of a signature?
- What are the possible weights?

The setup is $(\pi, \mathcal{H})$, an irreducible PER of LG at level l. So, π is a projective representation of $LG \rtimes \mathbb{T}_{rot}$, and we have an honest representation of $\mathbb{T}_{rot}$. There is a decomposition

$$\mathcal{H} = \bigoplus_{n \geq 0} \mathcal{H}(b)$$

with $\dim \mathcal{H}(n) < \infty$ for all n, and $\mathbb{T}_{rot}$ acts on $\mathcal{H}(n)$ by

$$z\xi = z^n \xi$$

or, equivalently

$$r_\theta(\xi) = e^{n\theta}\xi.$$

The classification is as follows:

Theorem. *For $(\pi, \mathcal{H})$ an irreducible PER, we have*

1. *$\mathcal{H}(0)$ is an irreducible $\mathfrak{su}(N)$-module for the embedding $SU(N) \subset LSU(N)$ by constant loops.*
2. *The signature of f of $\mathcal{H}(0)$ satisfies*
$$f_1 - f_N \leq l$$
3. *Conversely, if f is such a signature, then there is an irreducible PER of LG with $\mathcal{H}(0) \cong V_f$.*
4. *This representation is unique up to isomorphism.*[1]

For $LSU(n)$ at level l, the possible highest weights of $\mathcal{H}(0)$ are as indicated in the previous picture.

Question. Why is $\mathcal{H}(0)$ invariant under $SU(N)$?

Answer. Recall that $L^{poly}\mathfrak{g}$ denotes the trigonometric polynomials with values in $\mathfrak{g}$. There exists a representation ρ of $L^{poly} \rtimes \mathbb{R}$ such that

1. π factors through ρ:
$$\pi(e^x) = e^{\rho(x)}$$
2. If $X(n) = \rho(e^{-int}X)$ for $X \in \mathfrak{g}$, we have
$$[X(n), X(m)] = [X.Y](n+m) + ml\langle X, Y\rangle$$
3. The action of $x \in \mathbb{R}$ is given by
$$\rho(x) = xD,$$
where D acts on $\mathcal{H}(n)$ by multiplication by the energy n.
4. We have the commutation relation
$$[X(n), D] = -nX(n).$$

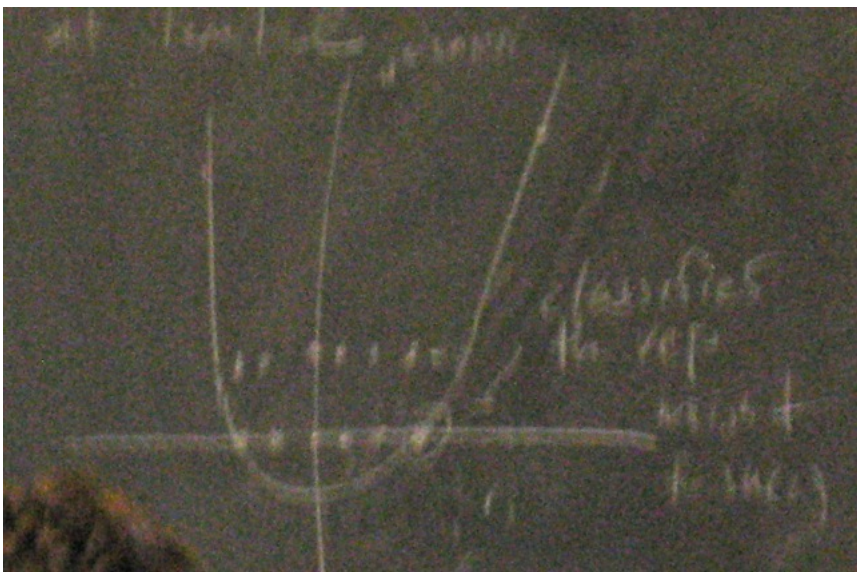
[1] If $\mathcal{H}$ and $\mathcal{H}'$ are two irreducible PERs of LG at level l such that $\mathcal{H}(0) \cong \mathcal{H}'(0)$ as $SU(N)$-modules, then there is a $U : \mathcal{H} \to \mathcal{H}'$ intertwining the projective actions of $LG \rtimes \mathbb{T}_{rot}$.

Then, for $\xi \in \mathcal{H}(j)$, we simply compute

$$\begin{aligned} DX(n)\xi &= X(n)D\xi - nX(n)\xi \\ &= (j-n)X(n)\xi. \end{aligned}$$

The $X(n)$ are analogous to the lowering operators that we employed when studying the representations of $\mathfrak{su}(N)$. In particular, taking $n = 0$, we see that $\mathcal{H}(j)$ preserved by constant elements of $L^{poly}\mathfrak{g}$, so that $\mathcal{H}(j)$ is an $SU(N)$-module via the embedding of constant loops. □

The next picture is the weight diagram for $LSU(2)$ at level l. In this picture, the vertical axis is the energy j, and the horizontal axis is the weight for $\mathfrak{su}(2)$:



The next question is where the third condition comes from. For the sake of Wassermann's paper, irreducible PERs, are sub-representations of $\mathcal{F}_P^{\otimes l}$, so $\mathcal{H}(0) \subset \mathcal{F}_P^{\otimes l}(0)$, and we should look for $SU(N)$-modules in $\mathcal{F}_P^{\otimes l}(0)$.

We want to find irreducible representations of $SU(N)$ that have signature $(f_1, f_1, \ldots, f_{N-1}, 0)$. Now,

$$\mathcal{F}_P = \bigwedge \mathcal{H}_P$$

where $\mathcal{H}_P = L^2(S^1, V)$ for $V = \mathbb{C}^N$. The space $\mathcal{H}_P$ contains a copy of V, embedded as constant functions, so in $\mathcal{F}_P^{\otimes l}(0)$, we have an embedded copy of $(\bigwedge V)^{\otimes l}$.

Recall that the element

$$e_1^{\otimes(f_1-f_2)} \otimes (e_1 \wedge e_2)^{\otimes(f_2-f_3)} \otimes \cdots \otimes (e_1 \wedge \cdots \wedge e_{N-1})^{\otimes(f_{N-1}-f_N)}$$

generates an irreducible $SU(N)$ module of signature f. Since

$$\left(\bigwedge V\right)^{\otimes l} = (\wedge^1 V \oplus \cdots \oplus \wedge^N V)^{\otimes l},$$

we can generate modules as before, provided that $f_1 \leq l$.