TOMITA-TAKESAKI THEORY AND THE KMS CONDITION

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Fixing notation: ϕ is a faithful, normal state on a von Neumann algebra M. What does that mean?

A state is a continuous linear functional $\phi : M \to \mathbb{C}$ such that $\phi(\mathbf{1}) = 1 = \|\phi\|$. Equivalently, $\phi(x^*x) \geq 0$ for all $x \in M$.

Faithful means that $\phi(x^*x) = 0$ if and only if $x = 0$.

Normal means that the state is continuous when M is given the ultraweak topology. That is, the topology wherre convergence is givenn by $x_\lambda \to x$ if

$$\sum_{i=1}^{\infty} \langle x_\lambda \xi_i, \eta_i \rangle \to \sum_{i=1}^{\infty} \langle x \xi_i, \eta_i \rangle$$

whenever

$$\sum \|\xi_i\|^2 + \|\eta_i\|^2 < \infty.$$

Equivalently, whenever x_λ is an increasing net in M that converges to x, we have $\phi(x_\lambda) \to \phi(x)$.

Put inner product on M by $\langle x, y \rangle = \phi(y^*x)$. This gives us a new norm, $\|\cdot\|_2$ on M. Complete M with respect to this norm to get the Hilbert space $L^2(M)$. Let Ω denote the image of $\mathbf{1}_M$ in $L^2(M)$. This is called the "vacuum vector."

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We let M act on $L^2(M)$, densely defined by left multiplication. For this action, Ω is a cyclic, separating vector for this action. Cyclic means $M\Omega$ is dense in $L^2(M)$, and separating means $x\Omega = 0 \implies x = 0$ for all $x \in M$.

We now look at the map $S_0 : M\Omega \to M\Omega$ given by $S_0(x\Omega) = x^*\Omega$. In generral, this map is unbounded, and so cannot be extended to $L^2(M)$. However, it is as nicely behaved as an unnbounded operator can be. We will be especially interested in its polar decomposition.

Can also define $F_0 : M'\Omega \to M'\Omega$ by $F_0(x'\Omega) = (x')^*\Omega$ for $x' \in M'$. Recall: M' is the commutant of M, as it acts on $L^2(M)$. i.e. $M' = \{x \in B(H) : xy = yx \text{ for all } y \in M\}$.

Notation: if A and B are unbounded operators, we write $A \subset B$ if $\operatorname{dom} A \subset \operatorname{dom} B$ and B agrees with A when restrict to $\operatorname{dom} A$.

If A is unbounded, the domain of A^* is $\{\eta : \langle A\zeta, \eta\rangle$ is a bounded function of $\zeta\}$. Then we have $\langle \zeta, A^*\eta\rangle = \langle A\zeta, \eta\rangle$ where the expression is defined.

Fact. $S_0 \subset F_0^*$ and $F_0 \subset S_0^*$.

Proof of fact. Let $a \in M$, $a' \in M'$. Then $\langle S_0(a\Omega), a'\Omega\rangle = \langle a^*\Omega, a'\Omega\rangle = \langle (a')^*\Omega, a\Omega\rangle$. This is clearly a bounded functino of $a\Omega$, which implies $\operatorname{dom} S_0 \subset \operatorname{dom} F_0^*$. The other inclusion is proved similarly. □

Since Ω is cyclic and separating for M, it is also cyclic and separating for M', so S_0 and F_0 are both densely defined. By a standard result in operator theory, both S_0 and F_0 are closable. That is, when we take the closure of the graph of S_0 or F_0, it remains the graph of a linear operator. These new operators have the continuity property that if $x_n \to x$ and $S_0(x_n)$ converges, then $S_0(x_n) \to S_0(x)$.

Theorem 0.1. *Let S and F be the closures of S_0 and F_0, respectively. Then $S = F^*$ and $F = S^*$.*

Since S and F are losed and densely defined, so we have a polar decomposition $S = J\Delta^{\frac{1}{2}}$. Recall that S is conjugate linear, so J is conjugate linear. In fact, J is an isometry. In general, Δ will be unbounded.

Claim. $J\Delta^{\frac{1}{2}}J = \Delta^{-\frac{1}{2}}$

Proof. $S = S^{-1}$, so $S^{-1} = (J\Delta^{\frac{1}{2}})^{-1} = \Delta^{-\frac{1}{2}}J$. Rearranging proves the claim. □

We now want to show $JMJ = M'$. We begin with some preliminary results.

Lemma 0.2. *With ϕ as before, and $\psi \in M_*$ such that $|\psi(y^*x)|^2 \leq \phi(x^*x)\phi(y^*y)$. Then $S = F^*$ and $F = S^*$ and given $\lambda > 0$ then there exists $a \in M$ with $\|a\| < \frac{1}{2}$ such that $\psi(x) = \lambda\phi(ax) + \lambda^{-1}\phi(xa)$.*

This is a sort of non-commutative Radon-Nikodym derivative.

Lemma 0.3. *Let λ be as before. Given $a' \in M'$, theere is an $a \in M$ such that $a\Omega \in \mathrm{dom}(F)$ and $a'\Omega = (\lambda S + \lambda^{-1}F)a\Omega$.*

Sketch of proof. From basic facts about the GNS construction, $\phi(x) = \langle x\Omega, \Omega\rangle$. We then have

$$\langle x\Omega, a'\Omega\rangle = \lambda\langle ax\Omega, \Omega\rangle + \lambda^{-1}\langle xa\Omega, \Omega\rangle$$

assuming $\|a\| < 1$. Once we check that everything is in the right domain, we get that

$$\lambda\langle ax\Omega, \Omega\rangle + \lambda^{-1}\langle xa\Omega, \Omega\rangle = \lambda\langle x\Omega, Sa\Omega\rangle + \lambda^{-1}\langle a\Omega, S(x\Omega)\rangle.$$

The last expression is $\lambda^{-1}\langle x\Omega, F(a\Omega)\rangle$. □

Lemma 0.4. *Let λ, a and a' be as before. Then if $xi, \eta \in \mathrm{dom}(F) \cap \mathrm{dom}(S)$, we have $\lambda\langle SaS\xi, \eta\rangle + \lambda^{-1}\langle FaF\xi, \eta\rangle = \langle a'\xi, \eta\rangle$.*

Proof.

$$F(0) = \int_{-\infty}^{\infty} \frac{F(it + \frac{1}{2}) + F(it - \frac{1}{2})}{2\cosh(\pi t)} dt$$

provided f is bounded, holomorphic on the strip $-\frac{1}{2} \leq \mathrm{Re}(z) \leq \frac{1}{2}$. □

Proposition 0.5. *Let λ, a and a' be as before. Then*

$$a = \int_{-\infty}^{\infty} \lambda^{2it} \frac{\Delta^{it} Ja'J\Delta^{-it}}{\cosh(\pi t)} dt$$

Proof idea. λ^{2it}*etc.* extend holomorphically, apply prev lemma □

Theorem 0.6. *$JMJ = M'$ and $\Delta^{it}M\Delta^{-it}M$ for all $t \in \mathbb{R}$.*

Proof idea. Take a unitary $u \in M'$. Then $a = u^*au$. Pull u under the integral given above, and note that the Fourier transforms of $u^*\Delta^{it}Ja'J\Delta^{-it}u$ and $\Delta^{it}Ja'J\Delta^{-it}$ are equal, so the operators are equal. Plugging in $t = 0$ impliess $JM'J \subseteq M$. By symmetry $JMJ \subseteq M'$, which proves the theorem. We used the fact that an operator commuting with every unitary in M' must commute with everything in M', and is therefore in M (by the double commutant theorem). □

Example. $M = M_2(\mathbb{C})$, and $\phi(x) = \mathrm{tr}(ax)$ where $a = \begin{bmatrix} \mu_1 & 0 \\ 0 & \mu_2 \end{bmatrix}$ for $\mu_i > 0$ and $\mu_1 + \mu_2 = 1$. In this example, $J(e_{ij}) = \sqrt{\frac{\mu_i}{\mu_j}} e_{ji}$ (extended by conjugate linearity) and $\Delta(e_{ij}) = \frac{\mu_i}{\mu_j} e_{ij}$ (extended linearly).

Example. Here we show an example where S is unbounded. Consider

$$M = \bigotimes_{i=1}^{\infty} M_2(\mathbb{C}).$$

Our state is $\phi = \bigotimes_{i=1}^{\infty} \phi_i$, where $\phi_i(x) = \mathrm{tr}(a_i x)$ as above with $\mu_{1,i} \to 0$ as $i \to \infty$ (and thus $\mu_{2,i} \to 1$).

Back to the $M_2(\mathbb{C})$ example. Fix $x, y \in M_2(\mathbb{C})$. Define $f(z)$ on the strip $0 \leq \operatorname{Im} z \leq 1$ by $f(z) = \langle \Delta^{-iz} y\Omega, x\Omega\rangle$. Then, for $t \in \mathbb{R}$, $f(t) = \phi(\sigma_t^\phi(x)y)$ and $f(t+i) = \phi(y\sigma_t^\phi(x))$. Here, $\sigma_t^\phi(x) := \Delta^{it} x \Delta{-it}$.

Theorem 0.7 (KMS condition)**.** *Define $f(z) = \langle \Delta^{-iz} x\Omega, y\Omega\rangle$ for $x, y \in M$ fixed. Then $f(t) = \phi(\sigma_t^\phi(x)y)$ and $f(t+i) = \phi(y\sigma_t^\phi(x))$. If α_t is a strongly continuous 1 parameter group of automorphisms of M satisfying $\phi \circ \alpha_t = \phi$ and there exists a function G, holomorphic in the strip, such that $G(t+i) = \phi(y\sigma_t^\phi(x))$ and $G(t) = \phi(\sigma_t^\phi(x)y)$, then $\alpha_t = \sigma_t^\phi$ for all $t \in \mathbb{R}$.*

Corollary 0.8. *The following are equivalent:*

(1) *$\phi(ax) = \phi(xa)$ for all $x \in M$*
(2) *$\sigma_t^\phi(a) = a$ for all $t \in \mathbb{R}$.*

Proof. (1) $\implies$ (2). If $x \in M$, then

$$\langle x^*\Omega, a\Omega\rangle = \langle \Omega, xa\Omega\rangle = \langle \Omega, ax\Omega\rangle = \langle a^*\Omega, x\Omega\rangle = \langle S(a\Omega), x\Omega\rangle.$$

This implies $a\Omega \in \operatorname{dom} S^*$ and $S^*(a\Omega) = a^*\Omega$. Sine $S^* = \Delta^{\frac{1}{2}} J$, $a^*\Omega$ has to be fixed by Δ. Hence $a\Omega$ is fixed by Δ and thus Δ^{it}.

(2) $\implies$ (1). We have $\phi(\sigma_t^\phi(x)a) = \phi(\sigma_t^\phi(xa)) = \phi(xa)$. This implies $f(z)$ is constant along the real axis, which implies that it is constant everywhere in the strip $0 \leq \mathrm{Im}(z) \leq 1$. Plugging in $t = 0$ we get $\phi(xa) = \phi(ax)$. ∎