TOMITA-TAKESAKI THEORY FOR FERMIONS

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ABSTRACT. Notes from the "Conformal Field Theory and Operator Algebras workshop," August 2010, Oregon.

Outline:

(1) Review of Tomita-Takesaki theory
(2) Examples
  (a) Modular Theory for fermions
  (b) Segal's CFT

1. Review

Notation: $L_p := L^{1/p}$. We change notation because these L's form a graded algebra, and lower indices indicate covariance (in topology).

Definition (Definition/Theorem)**.** If M is a von Neumann algebra, $L_*(M)$ is a $\mathbb{C}_{Re\geq 0}$-graded complex unital *-algebra, with maps $L_p(M) \times L_q(M) \to L_{p+q}(M)$ and $* : L_p(M) \to L_{\bar{p}}(M)$.

$L_0(M) \simeq M$ as *-algebras, and $L_1(M) \simeq M_*$ – canonically isomorphic to the predual. What's more, these are isomorphic as bimodules. (Analog to the Riesz lemma from functional analysis, $(L_1)^* \simeq L_0$.) (Bimodule structure on predual: $f \in M_*$, $m, x \in M$: $(mf)(x) := f(xm)$.)

There is also a trace $tr : L_1 \to \mathbb{C}$ such that $tr(xy - yx) = 0$.

$z \in L_p^+$, $p \in \mathbb{R} \iff$ there exists y, $y^*y = z$. If $p \in \mathbb{C}_{Re\geq 0}, q \in \mathbb{R}_{\geq 0}$.

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$$L_q^+(M) \to L_{qp}(M)$$
$$z \mapsto z^p$$

and if $p \in \mathbb{R}_{>0}$ then $L_q^+ \to L_{qp}^+$ is a bijection.

If the real part of p is zero, then the last map can be extended to unbounded measures and their powers: $\hat{L_q^+} \to L_{pq}$; elements $\phi \in \hat{L_1^+}$ are called *weights*

Question. How do you define these L_p for complex p?

Answer. If M is commutative, choose some measure μ. $L_p(\mu) := \{f | \int |f|^{1/Re(p)} < \infty\}$ if $Re(p) > 0$, or equal to the set of bounded functions if $Re(p) = 0$.

Definition. The *modular automorphism group: M* a von Neumann algebra, $\phi \in \hat{L_1^+}(M)$, $t \in \mathbb{I} := \{x \in C | Re(x) = 0\}$. Then $\sigma_t^\phi(x) = \phi^t x \phi^{-t}$, $\sigma_t^\phi \in Aut(L_p(M))$

$\sigma_s^\phi(xy) = \phi^s xy\phi^{-s} = \phi^x x\phi^{-s}\phi^s y\phi^{-s} = \sigma_s^\phi(x)\sigma_s^\phi(y)$ so it's a homomorphism.

Also easy to show $\sigma_s^\phi(\sigma_t^\phi(x)) = \sigma_{s+t}^\phi(x)$.

Definition. *Radon-Nikodym derivative* $\phi, \psi \in L_1^+$ and $t \in \mathbb{I}$. $(D\phi : D\psi)_t = \phi^t\psi^t \in L_0$. Note that the imaginary power makes unbounded things, bounded.

Theorem 1.1. *(KMS condition) Kubo-Martin-Schwinger*

For any M, there is a bijection between weights $\phi \in \hat{L_1^+}(M)$, and continuous one-parameter groups of elements in L_t, $t \in \mathbb{I} \mapsto U(t) \in L_t(M)$ such that $U(s+t) = U(s)U(t)$ and $U(s)^* = U(-s)$. The isomorphism is $U(t) = \phi^t$.

2. Examples

1. Suppose H is a $\mathbb{C}$-hilbert space and K is a closed real subspace such that $K \cap iK = 0$ and $K + iK$ is dense in H. $Cl^{alg}(K)$ acts on ΛH (a Hilbert space) by the creation and annihilation operators. $Cl(K)$ is the von Neumann algebra generated by $Cl^{alg}(K)$. $Cl(K^\perp)$ acts on ΛH; This makes ΛH a $Cl(K)$, $Cl(K^\perp)$ bimodule. Each of these is actually the commutant of the other on ΛH, whence $\Lambda H \simeq L_{1/2}(Cl(K))$. The vacuum vector $\Omega \in L_{1/2}(Cl(K))$ gives a finite weight by letting $\Omega = \phi^{1/2}$, $\phi \in L_1^+$.

For example, if $H = L_{1/2}(M)$ and $\phi \in \hat{L_1^+}$, let $K = \overline{M_{sa}\phi^{1/2} \cap L_{1/2}(M)}$. $\Delta^t \in Aut(L_{1/2}(M)) = Aut(H)$, and $* \in \tilde{Aut}(L_{1/2}(M)) = \tilde{Aut}(H)$.

Theorem 2.1 (Jones-Wasserman). (1) *ΛH is an invertible bimodule in the category of bimodules; the left and right actions are commutants of each other.*

(2) *$* : L_{1/2}(Cl(K)) = \Lambda H \circlearrowright$. For $\psi \in \Lambda H$, $\psi = a \wedge b \wedge c \wedge \cdots \wedge z$, $\psi^* = z^* \wedge \cdots \wedge a^*$.*

(3) *$\sigma_t^{\phi} : \Lambda H \circlearrowright$. $\psi \in \Lambda H$, $\psi = a \wedge b \wedge \cdots \wedge z$: $\sigma_t^{\phi}(\psi) = \sigma_t(a) \wedge \cdots \sigma_t(z)$.*