CONFORMAL NETS

SPEAKER: CORBETT REDDEN
TYPIST: PAVEL SAFRONOV

1. More Möbius group

It is the group of conformal automorphisms of $D \subset \mathbb{C}$. It is denoted by $PSU(1,1) \cong PSL_2(\mathbb{R})$.

$$S^1 - \text{picture} \quad \mathbb{R} - \text{picture}$$
$$z \leftrightarrow x$$
$$SU(1,1) \leftrightarrow SL_2(\mathbb{R})$$

It consists of translations $T_t x = x + t$, dilations $D_t x = e^{-2\pi t} x$ and rotations $R_t z = e^{-2\pi i t} z$.

Fact: we have an isomorphism of manifolds $SU(1,1) \cong D \times T \times R$.

Corollary: $SU(1,1)$ acts transitively on $\{I\}_{I \subset S^1}$, and

- isotropy group of $z \in S^1$ is $D \times T$.
- isotropy group of $\{z_1, z_2\} \in S^1$ is D.

2. Definitions of conformal nets and examples

Definition. A (vacuum) conformal net, denoted by CN (VCN), is a collection of von Neumann algebras $\{\mathcal{A}(I)\}_{I \subset S^1}$, parametrized by open, connected, non-dense intervals, that satisfy the following axioms

(1) (Isotony) $I \subset J \Rightarrow \mathcal{A}(I) \subset \mathcal{A}(J)$.
(2) (Locality) $I \subset J' = S^1 \backslash \overline{J} \Rightarrow \mathcal{A}(I) \subset \mathcal{A}(J)'$.

Date: August 18, 2010.
Available online at `http://math.mit.edu/~eep/CFTworkshop`. Please email `eep@math.mit.edu` with corrections and improvements!

(3) (Möbius covariance) There exists a representation $PSU(1,1) \to U(H)$, such that $\pi(g)\mathcal{A}(I)\pi(g)^* = \mathcal{A}(gI)$.
(4) (Positive energy) $R \subset PSU(1,1)$ should be positive energy.
Vacuum nets also satisfy:
(5) (Vacuum) There exists a unique (up to a factor) vacuum vector $\Omega \in H$, Ω invariant under the Möbius action and $\{\bigcup_{I \subset S^1} \mathcal{A}(I)\}'''\Omega$ is dense in H.

Remark: although $PSU(1,1)$ acts projectively, $U(1) \subset PSU(1,1)$ acts honestly.

Example: $\pi : \tilde{LG}_\ell \to U(H)$ be an IPER $\mathcal{A}(I) = \pi(\tilde{L_I G})''$.

Definition. An irreducible representation is called a vacuum representation if it has a vacuum Ω invariant under $PSU(1,1)$.

Theorem 2.1. *$\mathcal{A}$ is a conformal net. If π is a vacuum representation, then $\mathcal{A}$ is a vacuum conformal net.*

Proof. (1) $I \subset J \Rightarrow \tilde{L_I G} \subset \tilde{L_J G} \Rightarrow \pi(\tilde{L_I G})'' \subset \pi(\tilde{L_I G})''$.
(2) $I \cap J = \emptyset$, then $[\tilde{L_I G}, \tilde{L_J G}] = 1 \Rightarrow \mathcal{A}(I)$ commutes with $\mathcal{A}(J)$.
(3) $PSU(1,1)$ acts conformally on $D \subset S^1$, therefore canonically implemented.
(4) True by assumption.
(5) True by definition.

□

3. Properties

Theorem 3.1 (Reeh-Schlieder)**.** *If $\mathcal{A}$ is a vacuum conformal net, then Ω is cyclic for each $\mathcal{A}(I)$: $\overline{\mathcal{A}(I)\Omega} = H$.*

Corollary: Ω is cyclic and separating for each $\mathcal{A}(I)$.

Proof. Ω is separating for $\mathcal{A}(I) \Leftrightarrow \Omega$ is cyclic for $\mathcal{A}(I)'$. But Ω is cyclic for $\mathcal{A}(I') \subset \mathcal{A}(I)'$. □

Summary: $I \to \mathcal{A}(I)$, we have a cyclic and separating Ω, so can use Tomita-Takesaki theory. $S_I(A\Omega) = A^*\Omega$. From this we get the modular operators:

$$J_I \mathcal{A}(I) J_I = \mathcal{A}(I)'$$
$$\Delta_I^{it} \mathcal{A}(I) \Delta_I^{-it} = \mathcal{A}(I)$$

Theorem 3.2. *$\mathcal{A}$ is a vacuum conformal net.*

- *All $\mathcal{A}(I)$ are type III_1 factors.*
- *If inequality (almost always satisfied), then $\mathcal{A}(I)$ is hyperfinite and there is a unique (up to an isomorphism) type III_1-factor.*

Question: why is it a factor?

Answer: it follows from the axioms of a conformal net, not true for higher dimensions.

Let $j_{S_+} \in SU_-(1,1)$ be the flip.

Theorem 3.3 (Geometric modular operators)**.** *$\mathcal{A}$ is a vacuum conformal net.*

(1) *π extends to $PSU_{\pm}(1,1) \xrightarrow{\pi} U_{\pm}(H)$ such that $J_{S_+} = \pi(j_{S_+})$.*
(2) *$\Delta_{S_+}^{it} = \pi(D_t)$.*

Proof. (1) Check homomorphism for D, R, T.
(2) Work with equivariance properties.

□

Theorem 3.4 (Haag duality)**.** *If $\mathcal{A}$ is a vacuum conformal net, then $\mathcal{A}(I') = \mathcal{A}(I)'$.*

Proof. Because of the Möbius covariance, it suffices to show only for S_+.

$J_{S_+} \mathcal{A}(S_+) J_{S_+} = \mathcal{A}(S_+)' = \pi(j_{S_+} \mathcal{A}(S_+) \pi(j_{S_+}) = \mathcal{A}(S'_+)$. □

4. Representations

Definition. A representation of a conformal net $\mathcal{A}$ on a Hilbert space H_π is a collection of representations $\{\pi_I\}_{I \subset S^1}$, $\pi_I : \mathcal{A}(I) \to B(H_\pi)$, such that

(1) (Consistency) $I \subset J \Rightarrow \pi_I = \pi_J \big|_{\mathcal{A}(I)}$.

(2) There exists a representation $\pi^m : PSU(1,1) \to PU(H_\pi)$. $\pi^m(g)\pi_I(-)\pi^m(g)^* = \pi_{gI}(\alpha_{g-})$. Here α is a conjugation using the Möbius representation on $\mathcal{A}$.
(3) Rotations in π^m are generated by a positive operator.

Question: is it true that $\mathcal{A}(I)$ and $\mathcal{A}(J)$ commute in the representation if I and J are disjoint?

Examples:

- Identity representation: $\pi(\mathcal{A}(I)) = \mathcal{A}(I)$.
- Let $\mathcal{A}_0$ be a vacuum conformal net of level ℓ IPER of LG. $\pi : \tilde{LG}_\ell \to U(H_\pi)$ an IPER. Obtain a representation of $\mathcal{A}_0$.

Since π_0, π are subrepresentations of $\pi^{\otimes \ell}$ = factor representation, then local equivariance property from Min's talk to guarantee the map $\pi_0(L_I G)'' \to \pi(L_I G)''$.