THE BISOGNANO-WICHMAN THEOREM & NETS ON $\mathbb{R}^4$

SPEAKER: CHRISTOPH SOLVEEN
TYPIST: EMILY PETERS

Abstract. Notes from the "Conformal Field Theory and Operator Algebras workshop," August 2010, Oregon. Revised and extended by Christoph Solveen.

(1) Minkowski space
(2) Axioms for QFT
(3) BW theorem & Physics

We'll talk about the history of how the study of conformal nets came about. They are studied because they are simpler than physics!

Bisognano-Wichman theorem: There is a geometric action of the modular operators on Minkowski space. (The 1dim version is what Corbett called "Geometric Modular Operators" in his talk.)

A note on units: Unless otherwise stated, we choose "natural units" where the velocity of light c and the reduced Planck's constant $\hbar$ are equal to one.

1. Space-time

We begin by discussing the arena in which special relativistic physics takes place: *Minkowski spacetime* M^4. It is the real manifold $\mathbb{R}^4$, equipped with a particular pseudo-Riemannian metric η with signature $(+,-,-,-)$. The latter is called *Minkowski metric* and will be defined below.

Some history on special relativity: In the 1830s, *Faraday* realized that points in space and time should be involved in carrying the effect of a perturbation of a physical system over a distance. This contrasts the situation in *Newton*'s mechanics of point particles and leads to the concept of *fields*: at each point

Date: September 13, 2010.
Available online at `http://math.mit.edu/∼eep/CFTworkshop`. Please email `eep@math.mit.edu` with corrections and improvements!

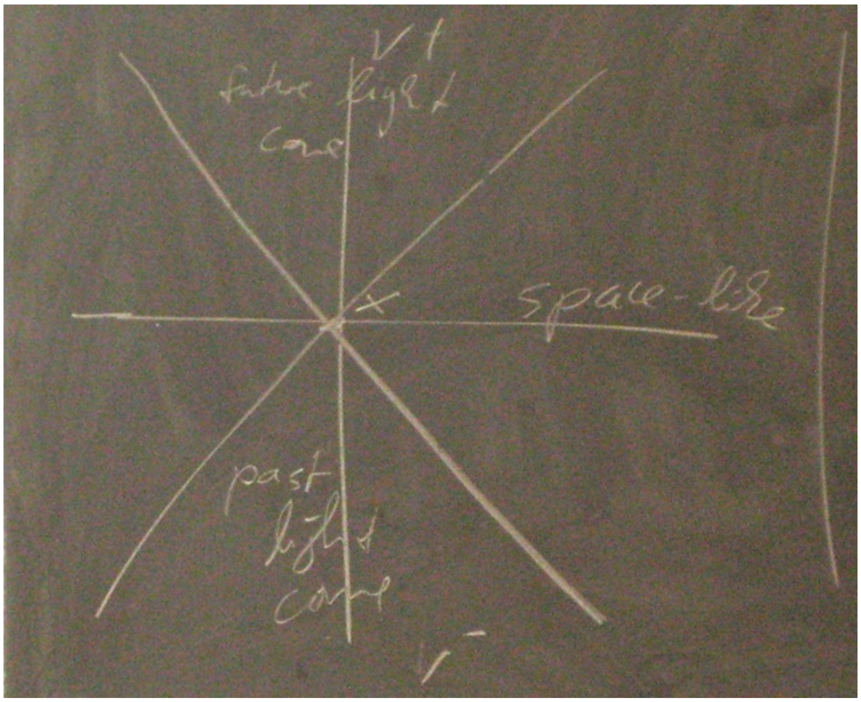
in space and time, one is given a set of observables, the field quantities. Mathematically, by a field one therefore means a section of a bundle over time $\times$ space.

Maxwell used this idea to write down his famous field equations for electromagnetism. They have (at least) two important properties: (1) The dynamics of the fields described by Maxwell is governed by a set of *hyperbolic* PDEs. This implies in particular that the effect of a perturbation of the field quantities propagates with finite velocity. In vacuo, this velocity is the speed of light $c = 3 \cdot 10^{10} m/s$. (2) Further the Maxwell equations are covariant under a group that had not made an appearance in physics before: the *Lorentz group*. In particular, the symmetry group of Newtonian physics, the *Galilei group*, which is supposed to relate between inertial observers (tied to observers that are not subject to any forces, i.e. "freely falling" ones), does not act covariantly on the fields. Following the Newtonian picture of space and time, this would mean that there is one preferred reference system in which the Maxwell equations hold while they do not hold in any other (this preferred coordinate system would be the rest system of the famous "ether").

It was *Einstein* who at the beginning of the 20th century rejected the idea of the ether and put more trust into the Maxwell equations than into the Newtonian concept of space and time: he promoted (1) (finite propagation speed for local perturbations of a physical system) and (2) (covariance of the laws of physics under the Lorentz group) to the status of axioms. This led to special relativity: by (1), the notion of a global time that is universal for all points in space looses its meaning. Rather, one needs a way of synchronizing clocks located at separated points and it follows that the division of the "arena" into time and space coordinates becomes observer-dependent. The class of inertial observers is not acted on by the Galilei group but rather by the Lorentz group.

Shortly after Einstein's great insights, it was the mathematician *H. Minowski* who realized that these ideas can be formalized as follows: we stop thinking about space and time separately but rather consider *spacetime*, a four-dimensional manifold diffeomorphic to $\mathbb{R}^4$. While we have lost an objective (coordinate-independent) notion of the division of this manifold into space and time, we should still be able to operationally define a notion of future, past and present of any given point ("event") in spacetime.

For this, one uses the finite speed of light: for any given event x one defines the *future light cone* L^+ to be the set of events that can be hit by light rays emanating from x. Its interior V^+ consists of events that can be hit by perturbations produced at x moving with speed less than c and forms the future of x. Likewise, the *past light cone* L^- consists of events that can hit x with a light signal. Its interior V^- is the set of events that can hit x with something that travels with speeds less than c and forms the past of x. The remaining events in spacetime that are not members of any of these sets form the present of x and are called *space-like separated* from x. They cannot exact any influence on x and vice versa.



Supposing the existence of a preferred class of coordinates on $\mathbb{R}^4$, the inertial coordinates, we introduce the metric η by requiring that its components in inertial coordinates are given by

$$\eta_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

With the help of this geometrical object, we can define the (square of the) *Lorentz distance* between $x, y \in \mathbb{R}^4$ to be $(x-y)^2 := \sum_{\mu,\nu} \eta_{\mu\nu}(x^\mu - y^\mu)(x^\nu - y^\nu)$. The light cone etc. described above is then formalized as follows:

$$\begin{aligned} L^+(L^-) \text{ of } x &:= \{y \in M^4 \,:\, (x-y)^2 = 0\,,\, x^0 - y^0 > (<) 0\} \\ V^+(V^-) \text{ of } x &:= \{y \in M^4 \,:\, (x-y)^2 > 0\,,\, x^0 - y^0 > (<) 0\} \\ \text{present of } x &:= \{y \in M^4 \,:\, (x-y)^2 < 0\}\,. \end{aligned}$$

For later purposes, we give the following

Definition. The *causal complement* O' of a subset O of Minkowski spacetime is given by

$$O' = \{y \in M^4 \mid (x-y)^2 < 0 \;\forall x \in O\}$$

(The unshaded region in the following picture). A region O is called *causally complete* if $O = O''$.

Events in O' cannot exact any influence on events in O (and vice versa). The basic example for causally complete regions are the double cones (or "diamonds") given by the interior of the intersection of the past light cone of x and the future light cone of y, where y is supposed to be in the past of x.

Inertial coordinates are related by the symmetry group of Minkowski spacetime, i.e. the group of diffeomorphisms of M^4 that leave the Lorentz distance invariant. This is the *Poincaré group*: $\mathcal{P} = \mathcal{L} \ltimes \mathbb{R}^4$. Here $\mathbb{R}^4$ is the group of translations of M^4 and $\mathcal{L}$ is the Lorentz group alluded to above. Interpreted as real 4×4 matrices acting on $\mathbb{R}^4$, the latter can be represented as $\mathcal{L} = \{\Lambda \in \mathrm{Mat}(n, \mathbb{R}) \;:\; \Lambda \eta \Lambda^T = \eta\}$. Since $\mathcal{L}$ still contains transformations that reverse spatial orientation and time-orientation, we restrict attention to its proper orthochronous subgroup $\mathcal{L}_+^\uparrow := \{\Lambda \in \mathcal{L} \;:\; \Lambda^0_0 > 1,\, \det(\Lambda) = 1\}$.

Correspondingly we have the proper orthochronous subgroup of the whole Poincaré group: $\mathcal{P}_+^\uparrow = \mathcal{L}_+^\uparrow \ltimes \mathbb{R}^4$.

There are two different subgroups of $\mathcal{L}_+^\uparrow$: namely the orientation preserving rotations of the spacelike slices of spacetime (which is just $SO(3)$) and the *Lorentz boosts* that relate inertial observers that are in uniform motion relative to each other. As an example of the latter (which is of relevance for the discussion later on), we consider a boost of an observer A to a frame of an observer B that travels into the x^1-direction of A with relative speed v. The boost is represented by

$$\Lambda(s) = \begin{pmatrix} \cosh(s) & -\sinh(s) & & \\ -\sinh(s) & \cosh(s) & & \\ & & 1 & \\ & & & 1 \end{pmatrix}, \tag{1}$$

where s is called the *rapidity* and is related to v by $\cosh(s) = (1 + \frac{v^2}{c^2})^{-\frac{1}{2}}$.

To summarize this section: we have given a very brief account of why we want to consider Minkowski spacetime as the arena for physics and introduced its geometry and discussed its symmetries that are physically thought of as tranformations between the coordinate systems (frames) used by a particular class of observers (the inertial ones).

2. Axioms for Quantum Field Theory

Before we lay down the axioms upon which quantum field theories (QFTs) are based from the point of view of AQFT, we wish to motivate why noncommutative algebras are used in their formulation.

What is physics? Very abstractly speaking, it is a pairing $(A, \omega) \mapsto \omega(A)$ where:

- A is an *observable*, i.e. a model for a measuring device,
- ω is a *state*, i.e. a way to prepare an ensemble of physical systems,
- $\omega(A)$ is a real number, the *expectation value* of measuring A in the ensemble represented by ω (i.e. the average value of the results of many measurements of A in ω).

We take $\omega(A)$ to be the expectation value of many measurements in an ensemble and not as a particular measurement result because we need to allow for statistical fluctuations: in general physics is a messy subject and we do not have full information about the system, the state is *mixed*, i.e. a convex combination of subensembles. However, as an idealization one may consider *pure* states which represent configurations of which we have optimal attainable knowledge.

If we model the set of observables by the self-adjoint elements of a unital $*$-algebra $\mathcal{A}$ (disregarding topological questions for a moment), the set of states is given exactly by the set of states on $\mathcal{A}$ in the mathematical sense, i.e. states are linear functionals ω on $\mathcal{A}$ which are normalized ($\omega(1) = 1$) and positive ($\omega(A^*A) \geq 0$). For positive linear functionals the Cauchy-Schwarz inequality holds:

$$|\omega(A^*B)|^2 \leq \omega(A^*A)\omega(B^*B),$$

which guarantees that the fluctuations

$$\Delta_\omega(A)^2 := \omega(A^2) - \omega(A)^2$$

are non-negative. Physically, $\Delta_\omega(A)$ is the standard deviation of many measurements of the observable A in the state ω. Pure states are taken as extremal elements of the (convex) set of states, i.e. elements of this set that cannot be represented as convex combinations of further states. Their existence is guaranteed if we specialize to C^*-algebras, which we will do from now on for the sake of simplicity.

We can now come to the dividing line between a) classical and b) quantum physics. a) In classical physics we expect no fluctuations in pure states. If $\mathcal{A}$ models the observables of a classical system, it is a commutative algebra (being the algebra of (smooth) functions on the phase space of the system). There is a theorem that says that the pure states of a commutative C^*-algebra are exactly the factorizing ones: $\omega(AB) = \omega(A)\omega(B)$. Therefore measurements in these states do not exhibit any fluctuations, they always yield the same result in all measurements, as expected. b) In quantum physics there are fluctuations even in pure states, optimal attainable knowledge does not mean that there is no uncertainty about the outcome of the measurement. To be honest, nobody can safely say why this is so. In quantum physics $\mathcal{A}$ is therfore taken to be non-commutative. Let us briefly explain the reasoning behind this: define the commutator of two elements A, B to be

$$[A, B] := AB - BA,$$

- it then follows from the Cauchy Schwartz inequality that for any state ω and any two elements $A, B \in \mathcal{A}$ we have

$$\Delta_\omega(A)\Delta_\omega(B) \geq \frac{1}{2}|\omega([A, B])|.$$

This is a way to express *Heisenberg*'s famous uncertainty principle. If two observables A, B do not commute, they are *non-commensurable*, i.e. we may find a state with really small fluctuations in A, but Heisenberg's uncertainty principle tells us that the fluctations in B have to be really large at the same time. This also tells us that even in pure states there will be observables that have non-vanishing fluctuations. Therefore we have to consider non-commutative algebras in quantum theory.

In the following, we will present axioms for *quantum field theory* (QFT). QFT was devised in the 30ies of the last century in order to bring together the principles of special relativity and quantum physics. Physically this means this theory should be able to describe quantum processes that take place in a regime where special relativistic effects are important, such as the collider experiments of elementary particle physics. Perturbative QFT does so with great success, as exemplified by the famous standard model of particle physics (this specific QFT model was written down in the early 70ies after people realized how useful (non-Abelian) gauge theory is). However, the mathematical structure of QFT has remained a mystery for decades, which motivated mathematical physicists to lay down physically clear axioms as starting points to investigate the theory with mathematical rigour and care. To this day, the structure of the standard model is not understood in this context, it can only be treated perturbatively. However, many conceptual questions and results concerning the general structure of QFT have been derived in the framework of rigorous QFT (called also algebraic QFT, local quantum physics or general QFT). We will now write down the *Haag-Kastler axioms* for nets of von Neumann algebras on Minkowski spacetime - which should look very similar to the ones for conformal nets on the circle that our workshop is mainly concerned with. However, even if they look similar and many of the results on S^1 you have seen so far have an analogue on $\mathbb{R}^4$ (in fact, they have most likely been proven in this framework first), the axioms for nets on $\mathbb{R}^4$ give rise to mathematics that is vastly more complicated.

Definition. *Nets of Observable algebras and Haag-Kastler axioms:* A net of von Neumann algebras on Minkowski spacetime, $\mathcal{O} \mapsto \mathcal{A}(\mathcal{O})$, where $\mathcal{O}$ are open, bounded, causally complete ($\mathcal{O} = \mathcal{O}''$) regions of M^4, satisfies the Haag-Kastler axioms if

(1) $\mathcal{O}_1 \subset \mathcal{O}_2 \Rightarrow \mathcal{A}(\mathcal{O}_1) \subset \mathcal{A}(\mathcal{O}_2)$ (Isotony)
(2) $\mathcal{O}_1 \subset \mathcal{O}_2' \Rightarrow \mathcal{A}(\mathcal{O}_1) \subset \mathcal{A}(\mathcal{O}_2)'$ (Locality)
(3) There is an automorphic action of $\mathcal{P}_+^\uparrow$ with $\alpha_g(\mathcal{A}(\mathcal{O})) = \mathcal{A}(g\mathcal{O})$ for $g \in \mathcal{P}_+^\uparrow$. (Poincaré Symmetry)

The *quasi-local* algebra $\mathcal{A}$ is defined as the C^*-completion of the union of all the algebras of the net.

Item (1), isotony, is required for consistency: if we use a measurement apperatus that is placed in some spacetime region, it is surely also placed in any spacetime region containing the original one. Item (2), locality, expresses the requirement that observables placed in spacelike separated regions should be commensurable in the sense explained above - their measurement fluctuations in any state should be independent of one another. You may have heard about non-local effects in quantum theory as expressed by the EPR-paradoxon and Bell's inequalities. These matters concern correlations among spacelike separated measurements and are not covered by this axiom. Item (3), (Poincaré Symmetry), requires that the action of $\mathcal{P}_+^\uparrow$ on the net is geometrical: it maps the algebra of one region onto the algebra of the transformed region, thereby preserving all algebraic relations.

So far, we have considered an abstract net only, without reference to the underlying Hilbert space. However, to arrive at the usual picture of quantum theory, where operators and Hilbert spaces play a prominent role, one has to discuss representations of the net. The simplest one, which is of relevance in particle physics, is the vacuum representation.

Definition. *The vacuum representation* A representation π_0 of the Haag-Kastler net $\mathcal{O} \mapsto \mathcal{R}(\mathcal{O})$ on some Hilbert space H is called *vacuum representation* if:

(4) There exists a strongly continuous unitary representation U of $\mathcal{P}_+^\uparrow$ that implements the Poincaré symmetry: $U(g)\pi_0(\mathcal{R}(\mathcal{O}))U(g)^* = \pi_0(\alpha_g(\mathcal{R}(\mathcal{O})))$. (Poincaré Covariance)
(5) The joint spectrum of the generators of the translation subgroup $U|_{\mathbb{R}^4}$ lies in V^+. (Spectrum Condition)
(6) There exists a unique translation invariant vector $\Omega \in H$ and the set $\{\pi_0(A)\Omega : A \in \mathcal{R}(\mathcal{O}), \mathcal{O} \text{ double cone}\}$ is dense in H. (Vacuum)

Item (4), Poincaré Covariance, realizes the Poincaré group as symmetry group of the system in the usual quantum mechanical sense. Item (5), the Spectrum Condition, ensures that every inertial observer measures positive energy (the corresponding observable is the generator of time translations, the "Hamiltonian") and constitutes a stability condition: roughly speaking, in the vacuum, the spectrum of the energy should have a lower bound, otherwise we could extract arbitrary amounts of energy for extended periods of time out of "nowhere". Item (6) ensures that there is a $\Omega \in H$ which formalizes our idea of the vacuum: it looks the same everywhere and is, in fact, the state with lowest (zero) energy. Under the stated assumptions, it follows that the vacuum Ω is Poincaré-invariant also - it looks the same to any inertial observer and to each such observer, it is "empty" - it has energy zero.

We mention the following remarkable result, which is known as "Reeh-Schlieder"-theorem.

Theorem 2.1. Reeh-Schlieder Theorem *If the Haag-Kastler net is additive, i.e.*

$$\mathcal{O} = \cup_i \mathcal{O}_i \;\Rightarrow\; \mathcal{R}(\mathcal{O}) = \vee_i \mathcal{R}(\mathcal{O}_i),$$

then the vaccum vector Ω *is cyclic and separating for any double cone.*

This is quite amazing, because one would like to think of some vector $A\Omega \in H$ with $A \in \mathcal{A}(\mathcal{O})$ as describing an excitation of the vacuum localized in the region $\mathcal{O}$. However, the theorem tells us that we can approximate $A\Omega$ to arbitrary precision by acting on the vacuum Ω with elements of $\mathcal{A}(\tilde{\mathcal{O}})$ with $\tilde{\mathcal{O}} \subset \mathcal{O}'$, i.e. by observables whose placement is spacelike separated to $\mathcal{O}$. So by making use of "vacuum fluctuations" here, we can in principle affect regions very far away. ("In principal, we could without leaving this room build a Taj Mahal behind the moon. ") However, such effects are in general surpressed, roughly because one would have to invest truely insane amounts of energy to make it happen. For us, the result is interesting mathematically, it says that for each double cone, the vacuum vector is standard for the local algebra associated to this region and we may define the corresponding Tomita-Takesaki objects (known from previous talks).

Note that there are other representations which are of relevance in physics, for example those describing "charged sectors" that are important for elementary particle physics. Another type, which we will come across in these notes, are representations describing a configuration that is in global thermal equilibrium at a given inverse temperature $\beta = \frac{1}{k_B T}$, where T is the absolute Temperature and k_B is Boltzmann's constant which we will put equal to one by a suitable choice of units. By the GNS construction, there is a one-to-one correspondence between states of a net and representations thereof. The equilibrium reps alluded to come about as GNS reps of so-called KMS states, which we already came across in earlier talks.

Definition. *KMS condition* Consider an algebra A that comes equipped with a one-parameter group of automorphisms $\{\gamma_t\}_{t\in\mathbb{R}}$. A state φ of A is called (γ_t, β)-KMS state for some $\beta \in \mathbb{R}$, if, for each pair $a, b \in A$, there is a function $h : \mathbb{C} \to \mathbb{C}$, analytic in the strip $\{z \in \mathbb{C} : 0 < \mathrm{Im}(z) < \beta\}$ and continuous at the boundaries, such that

$$h(t) = \varphi(a\,\gamma_t(b)) \;\text{ and }\; h(t + i\beta) = \varphi(\gamma_t(b)\,a)$$

for $t \in \mathbb{R}$.

It is a consequence of the definition that the state φ is invariant under γ_t for all $t \in \mathbb{R}$. The physical relevance of the KMS-condition now comes from the fact that it is the suitable generalization of *Gibbs ensembles* (that describe thermal equilibrium situations in quantum statistical mechanics of finite systems) to the infinitely extended medium, like a quantum field in Minkowski spacetime. We will not give any details here.

What is important for us here is the following fact: Consider a state ω of the net. If the time evolution (i.e. the dynamics) of an observer[1] is given by the one-parameter group of automorphisms $\{\gamma_t\}_{t\in\mathbb{R}}$ of the quasilocal algebra $\mathcal{A}$, and if ω is an (γ_t, β) KMS-state with $\beta \in \mathbb{R}^+$, then ω represents an ensemble which is in global thermal equilibrium at inverse tmeperature β for this observer. As was mentioned right after the definition of the KMS condition, the state is invariant under the dynamics (to the observer, it looks the same at all times). One finds that the time translations can be unitarily implemented and their generator is again a measure for the energy. However, its spectrum is now unbounded, there is no lowest energy - simply because you can extract arbitrary amounts of energy from an infinite "hot" medium.

There is another set of axioms which is closer in spirit to the traditional field theoretic approach that physicists are used to. It is called the *Gårding-Wightman axioms.* Its main objects are not algebras of observables but quantum fields, represented as operator valued distributions. Recall that classically, a field is (locally) a map that assigns to each point in spacetime a set of observables, the field quantities. In quantum theory, observables are modelled by elements of some non-commutative $*$-algebra, which can be represented by (possibly unbounded) operators in some Hilbert space. Therefore, we can try to "quantize" the classical idea of a field by requireing that a quantum field is a map from spacetime into the set of operators of some Hilbert space (operator valued function). However, this map must satisfy some properties that are motivated by physics: locality, covariance under the Poincaré group, some stability condition etc. . It turns out that the resulting map cannot be an operator valued function because it is too singular, we must resort to operator valued distributions, i.e. maps from some test function space (usually the Schwartz class functions $\mathcal{S}(\mathbb{R}^4)$) into the set of unbounded operators of some Hilbert space. More precisely, we have the following axioms, which for simplicity we will state only in the case of a "scalar, Bosonic field".

Definition. *Quantum Fields and Gårding-Wightman axioms* A Quantum field is a linear map $f \mapsto \phi(f)$ from $\mathcal{S}(\mathbb{R}^4)$ into the set of (unbounded) operators on a Hilbert space H such that

(1) there is a dense invariant domain $D \subset H$ for all fields $\phi(f)$. $\phi(\overline{f}) \subset \phi(f)^*$ and for real-valued f, the $\phi(f)$ are essentially self-adjoint.
(2) there is a strongly continuous representation U of the Poincaré group $\mathcal{P}_+^\uparrow$ with $U(g)\phi(f)U(g)^* = U(f_g)$ (here $f_g(x) := f(g^{-1}(x))$),
(3) the joint spectrum of the generators of $U|_{\mathbb{R}^4}$ (the translations) lies in V^+,

[1]The time evolution of an inertial observer is given by the one-parameter group of automorphisms given by his time-translations as in item (3) of the Haag-Kastler axioms.

(4) there is a unique (up to phase) translation invariant vector $\Omega \in D$, the "vacuum" vector,
(5) if the supports of f and g are spacelike separated, then $\phi(f)\phi(g) = \phi(g)\phi(f)$ on D.

One may ask about the connection between the two sets of axioms. While things are not perfectly clear, there are some results. First of all, given a set of Wightman fields on some Hilbert space H, one may arrive at a Haag-Kastler net with vacuum representation H by defining $\mathcal{A}(\mathcal{O})$ to be the von Neumann algebra generated in $B(H)$ by the bounded functional calculi of the fields $\phi(f)$ with $\text{supp}(f) \subset \mathcal{O}$. For the other direction, i.e. determining the field content of a net, one starts from the idea that fields, in some sense, are pointlike localized quantities and one should therefore be able to extract the field observable at $x \in M^4$ as member of $\bigcap_{\mathcal{O}\ni x} \mathcal{A}(\mathcal{O})$. However, this set turns out to be trivial: it consists only of multiples of the unit. One can modify this idea and really arrive at a meaningful field content if the theory has a nice "high energy behaviour" [2].

There are a number of interesting results concerning the structure of any Wightman QFT like the famous Spin-Statistics Theorem, but we will mention here only the following two, because we need them later. Again, we have the Reeh-Schlieder-theorem[3], which is interesting for the same reasons as before. Nomenclature: a Wightman QFT is called *irreducible* if apart from multiples of the identity there is no bounded operator in $B(H)$ that commutes with all fields $\phi(f)$.

Theorem 2.2. Reeh-Schlieder Theorem *Assume the Wightman field ϕ is irreducible. Then, for any non-empty open region $\mathcal{O}$ of Minkowski spacetime, the vacuum vector Ω is cyclic for the von Neumann algebra $\mathcal{A}(\mathcal{O})$ associated to the fields $\phi(f)$, f supported in $\mathcal{O}$ (in the sense described beore). By locality, it is standard (i.e. also separating) for $\mathcal{A}(\mathcal{O})$ if $\mathcal{O}'$ is non-empty.*

It was one of the early successes of rigorous QFT to show that in any Wightman QFT exhibits a particular kind of symmetry, the PCT ("parity-charge-time") symmetry. More precisely

Theorem 2.3. (part of) the PCT Theorem *For a Wightman field there exists an anti-unitary operator Θ uniquely defined by:*

$$\Theta\phi(f)\Theta^{-1} = \phi(f_-)^* \ ; \ \Theta\Omega = \Omega,$$

where $f_-(x) := f(-x)$.

[2] By the uncertainty relations, one has to invest higher and higher energies in order to measure in increasingly smaller regions, so imagine what happens if you try look for observables that measure at a point ...

[3] In fact, Reeh and Schlieder proved it in the Wightman setting long ago, while the corresponding result in the Haag Kastler setting is much more recent work by Borchers.

Θ implements a reversal of the spatial and time coordinates ("P" and "T") and flips the "charge" of the field ("C") at the same time. We need Θ for the Bisognano Wichmann theorem below.

3. Bisognano-Wichmann theorem and Physics

In this section, we are going to state the Bisognano-Wichmann theorem (BW theorem) and discuss some of its consequences. Originally, the BW theorem was proven in the Wightman framework. It has not been proven in general for Haag-Kastler nets, but there has been progress in the recent years which I mention in the beamer slides for this talk. Check there if you're interested. In particular, if the Haag-Kastler net possesses is a larger symmetry group than usual (i.e. the conformal symmetry group on $\mathbb{R}^4$) then you do get the BW property.

We need to consider so-called wedge regions of Minkowski spacetime, for example the right wedge in x^1-direction:

$$W_R = \{x \in \mathbb{R}^4 : x^1 > |x^0|\}.$$

All other wedges are obtained by shifting W_R: $g(W_R)$ for $g \in \mathcal{P}$. Note however that W_R is invariant under the 'boosts' given in equation (1): $\Lambda(s)W_R \subseteq W_R$ for all $s \in \mathbb{R}$. Given a Gårding-Wightman QFT, we consider the corresponding Haag-Kastler net $\mathcal{O} \mapsto \mathcal{A}(\mathcal{O})$ that is associated to it in the sense described in the previous section. Denote by $\mathcal{A}(W_R)$ the von Neumann algebra generated by the bounded functional calculi of the fields $\phi(f)$ with $\mathrm{supp}(f) \subset W_R$. Assuming that the field is irreducible, we know that by the Reeh-Schlieder Theorem 2.1 the vacuum vector Ω is standard for $\mathcal{A}(W_R)$ and we find the corresponding Tomita-Takesaki modular objects J_W, the modular conjugation, and Δ_W, the modular operator that gives rise to the modular group

$$\sigma_\tau^W(\cdot) := \Delta_W^{i\tau}(\cdot)\Delta_W^{-i\tau}$$

acting on $\mathcal{A}(W_R)$.

The BW Theorem now states that, remarkably, the action of these objects on elements of the algebra is geometric. Denote by $U(R_{23}(\cdot))$ and $U(\Lambda(\cdot))$ the unitary implementations of the rotations in the (x^2, x^3)-plane and the boosts $\Lambda(\cdot)$ (as in equation (1)) respectively. Moreover, we define the one-parameter subgroup

$$\gamma_s(\cdot) := U(\Lambda(s))(\cdot)U(\Lambda(s)^*$$

corresponding to the action of the boosts on $\mathcal{A}(W_R)$.

Theorem 3.1. Bisognano-Wichmann Theorem *Under the named assumptions, we have:*

$$J_W = \Theta \cdot U(R_{23}(\pi)) \quad \textit{and} \quad \Delta_W^{i\tau} = U(\Lambda(-2\pi\tau)) \textit{ for all } \tau \in \mathbb{R}.$$

Therefore $\sigma_\tau^W = \gamma_{-2\pi\tau}$. *Here,* Θ *is the PCT-operator introduced in the previous section.*

Note that since the Poincaré group acts geometrically on the net, we have similar results for all other wedges as well.

Before we can discuss one very interesting consequence of this theorem for physics (i.e. the Unruh effect), we need to mention the relationship between modular theory and the KMS condition/thermal equilibrium (which has a very interesting history in its own right).

Proposition 3.2. *Let* φ *be a state of a von Neumann algebra* A *with corresponding GNS rep* π_φ *and cyclic GNS vector* Ω_φ *that is also separating for* $\pi_\varphi(A)''$ [4]. *Denote the corresponding modular group by* $\{\sigma_\tau\}_{\tau\in\mathbb{R}}$. *Then* φ *is a* $(\sigma_\tau, -1)$*-KMS state.*

Taken together with the characterization of global thermal equilibrium situations via the KMS property, this proposition has the following important consequence in the Haag-Kastler setting.

Corollary 3.3. *A global thermal equilibrium state with respect to an observer with time evolution* α_t *(t physical time, i.e. the* proper time *of the observer) with inverse temperature* β *may be characterized as a state over the quasi-local algebra whose modular automorphism group* σ_τ *(*τ *is the "modular parameter") is related to to* α_t *by*

$$\sigma_\tau = \alpha_{-\beta\tau} \ ,$$

i.e. $t = -\beta\tau$.

In order to explain what this has to do with the BW theorem, we need to discuss the geometrical setup a bit more. We will be very brief, otherwise we would have to go into too much special relativity. First of all, note that the orbit of the point $x = (0, a^{-1}, 0, 0) \in W_R$, $a \in \mathbb{R}^+$, under the boosts $\Lambda(\cdot)$ (equation (1)) is precisely the trajectory of a uniformly accelerated observer with acceleration a (i.e. the acceleration along the curve $\Lambda(s)x$ is given by a). Moreover, the proper time t of the observer, i.e. the time measured by a clock that he or she carries around[5], is given by $t = a^{-1}s$.

[4]This is the case if φ is faithful (i.e. $\varphi(a^*a) = 0 \Rightarrow a = 0$), for example.

[5]Essentially, proper time is just the arc length of the curve in Minkowskian geometry. It is positive for physical observers, i.e. for observers moving with a velocity of less than c.

This tells us - going back to the notation introduced before the BW theorem - that the time evolution of this particular observer on $\mathcal{A}(W_R)$ is given by the one-parameter group of automorphisms $\alpha_t := \gamma_{at}$. By the BW theorem, we therefore know that

$$\alpha_t = \sigma^W_{-\frac{a}{2\pi}t}.$$

On the other hand, Corollary 3.3 tells us that

$$\alpha_t = \sigma^W_{-\frac{1}{\beta}t}$$

and hence we find $\beta = \frac{2\pi}{a}$ or, in terms of the absolute temperature:

$$T = \frac{a}{2\pi} \ . \tag{2}$$

This is the famous *Unruh Temperature.* This result is quite astonishing: it says that an observer in uniformly accelerated motion perceives the vacuum (which for an inertial observer is "empty") as a heat bath with temperature proportional to his or her acceleration. If you think about it, this is really amazing. However, this effect is incredibly small, going back to *cgs* units, one finds that an acceleration of $1g$ gives rise an Unruh Temperature of the order of magnitude of 10^{-20} degrees Kelvin.

This effect was discovered by Unruh[6] in an attempt to understand the *Hawking effect.* The Hawking effect is the emission of thermal radiation by a black hole that interacts with a quantum field. Classically, a black hole is truly "black", it does not emit anything - but *Hawking* made the great discovery that once you let a *quantum* field propagate on a black hole spacetime, it actually starts to emit thermal radiation, i.e. it aquires a temperature, the *Hawking Temperature* that looks similar to the Unruh Temperature. This and many other interesting questions are the subject of "Quantum Field Theory on Curved Spacetime", which investigates the effects that spacetime curvature (as in general relativity) has on quantum field theory and thereby makes heavy use of algebraic QFT. Incidentally, this is where my own research is located.

[6]Only later it was observed that the BW theorem establishes this effect rigorously in any QFT model.