MORNING DISCUSSION

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Abstract. Notes from the "Conformal Field Theory and Operator Algebras workshop," August 2010, Oregon.

There were some questions about the physics motivation for what we're doing.

Minkowski space

"everything" – H_0.

Quantum fields. So, what is a quantum field? It's an perator valued distribution – a map from smooth functions on Minkowski space to $End(H_0)$. If it was bounded maps it would be nice, but in typical examples its' not. This should be linear, although that's difficult to define.

So, a quantum field theory is a choice of H_0 and a bunch of fields. These fields are subject to some relations w.r.t. "composition" – ie a multiplication in $End(H_0)$, which ipso facto is not well-defined. This multiplication might be "operator product expansion?" (Audience member: No, it's not. This is the Liebniz setting – OPE is unnecesary/innapropriate here.)

This formalism is called "Wightman fields."

We would like to take $\mathbb{R}^4$ as our Minkowski space – but this is hard to work with. Good examples are not known.

So we try an easier case – $\mathbb{R}^2$ is the smallest spacetime.

Symmetries of Minkowski space: Poincare group including translations, rotations. We also need to Poincase group to act on H_0; so It acts on both

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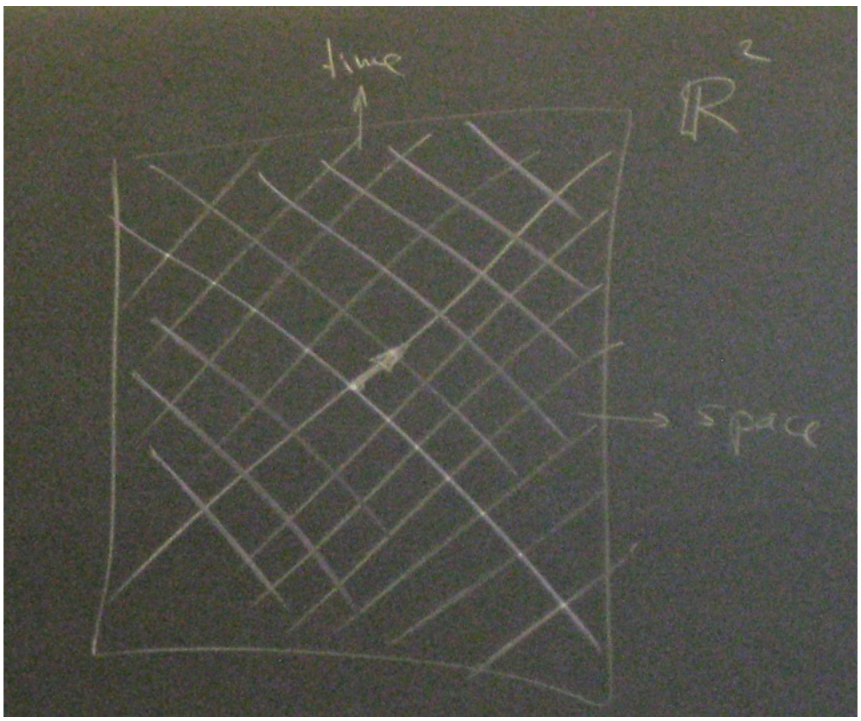
sides of the operator valued distribution (smooth function on Minkowski space, and $End(H_0)$.

For *conformal field theory*, we add more symmetries: look at conformal diffeomorphisms of $\mathbb{R}^2$. (conformal means the metric is sent to a function times the metric).

We have the nice formula $Conf(Mink) = Diff_+(\mathbb{R}) \times Diff_+(\mathbb{R})$. Why?

we have distinguished light rays in $\mathbb{R}^2$; these give us foliations.



By defn of conformal, this foliation is preseved by any conformal map. So our only freedom is to reparametrize in the NE or the NW directions.

A *chiral* conformal field theory is a QFT on $\mathbb{R}$ with symmetry group $Diff_+(\mathbb{R})$.

Last step: replace $\mathbb{R}$ by $\mathbb{R} \cup \{\infty\} = S^1$, for the sake of compactness.

Caution: anomolies in $Diff(S^1)$, ie it doesn't really act on H_0 but it acts projectively.

Example. Let's see an example of a field.

In a Dirac field, generated by two fields ψ and ψ^*, we have $\psi(z)$ ($z \in S^1$ for now) subject to relations

$$[\psi(z), \psi(w)]_+ = 0$$
$$[\psi^*(z), \psi^*(w)]_+ = 0$$
$$[\psi(z), \psi^*(w)]_+ = \delta(z - w)$$

with $[a, b]_+ = ab + ba$.

Now, let H_0 be Fock space $\mathcal{F}_P = \Lambda(PH) \otimes \Lambda(PH^\perp)^*$ with $H = L^2(S^1)$ and $PH=$ Hardy space, namely the (closure of the) span of things of the form z^n with $n \geq 0$.

Now,

$$\psi : \{L^2 \text{ functions on } S^1\} \to B(H_0)$$
$$f \mapsto \psi(f) := a(f)$$

With $a(f)$ being yesterday's creation operator. $a(f)$ satisfies $[a(f), a^*(g)]_+ = \langle f, g\rangle$.

Formally, $\psi(z) = \psi(\delta_z)$ with $a(f) = \int f(z)\psi(z)$. This is a "smeared field." We can check that this still satisfies the right commutation relations.

i.e.,

$$\begin{aligned} [a(f), a^*(g)]_+ &= [\int f(z)\psi(z), g(w)^*\psi^*(w)] \\ &= \int\int f(z)\overline{g(w)}[\psi(z), \psi^*(w)]_+ = \int f(z)\overline{g(z)} \end{aligned}$$

Note: Fields that go with $SU(N)$ are "currents." Good luck writing down explicit commutation relations for smeared currents.

What do the computations look like here? $\mathfrak{g}$, $\{X_\alpha\}$ a basis of $\mathfrak{g}$, and $[X_\alpha, X_\beta] = c^\gamma_{\alpha\beta}X_\gamma$. The nonabelian currents are

$$[J_\alpha(z), J_\beta(w)] = \Sigma_\gamma c^\gamma_{\alpha\beta} J_\gamma(w)\delta(z - w) + \ell \langle X_\alpha, X_\beta\rangle \frac{\partial}{\partial w}\delta(z - w)$$

.

$J_\alpha(f) = \int f(z)J_\alpha(z)$. Now $f(z)X_\alpha \in L\mathfrak{g} = C^\infty(S^1, \mathfrak{g})$.