THE HYPERGEOMETRIC FUNCTION

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ABSTRACT. Notes from the "Conformal Field Theory and Operator Algebras workshop," August 2010, Oregon.

Want to relate F_μ and G_μ after analytic continuation. Writing F_μs in terms of G_νs – coefficients are "transport coefficients."

(1) Hypergeometric function/equation
(2) Compute transport coefficients for the "Basic ODE"

Definition. *Gauss's hypergeometric equation:* second order ODE with 3 regular singular points $\{0, 1, \infty\}$:

$$z(1-z)f'' + [c - (1+a+b)z]f' - abf = 0.$$

What's cool about this are its solutions, built from

$$_2F_1\,(a, b; c; z) = \Sigma_{n\geq 0} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!}$$

with $(a)_n := a(a+1)\cdots(a+n-1)$.

Rewrite differential equation as

$$F'(z) = (\frac{A}{z} + \frac{B}{1-z})F(z)$$

with $A = \begin{pmatrix} 0 & 1 \\ 0 & 1-c \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 0 \\ -ab & 1+a+b-c \end{pmatrix}$ and $F = \begin{pmatrix} f(z) \\ zf'(z) \end{pmatrix}$

Replacing A, B by n-by-n matrices, this same equation is called an *abstract hypergeometric equation*. Here $(a)_n = \frac{\Gamma(a+n)}{\Gamma(n)}$ with $\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt$; for natural numbers x, $\Gamma(x) = x!$.

Notation and assumptions: $f'(z) = \frac{Pf}{z} + \frac{Qf}{1-z}$, $f : \mathbb{C} \to V = \mathbb{C}^n$, $P, Q \in \mathrm{End}(V)$. P has eigenvalues λ_i with differ mod 1, and eigenvectors ξ_i. Q is a

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nonzero multiple of a rank-one idempotents, so $Q^2 = \delta Q$ for some nonzero δ; so $tr(Q) = \delta$. Rank one implies that there's some $\phi \in V^*$ and $v \in V$ so that $Q(x) = \phi(x)v$, $\phi(v) = \delta$.

Q is in general position w.r.t. P: $v = \Sigma \delta_i \xi_i$, with $\delta_i \neq 0$.

Choose eigenvectos so that $\phi(\xi_i) = 1$. Let $R = Q - P$ and suppose R satisfies the same conditions as P with respect to Q. Let $(\zeta_j, -\mu_j)$ be the normalized eigenvectors/values.

Look at power series for the solutions around zero, call these f_i: $f_i(z) = \Sigma_n \xi_{i,n} z^{\lambda_i + n}$; $\xi_{i,0} = \xi_i$. This converges in $\{z : |z| < 1, z \notin [0, 1)\}$.

Similarly the solutions around infinity, g_j: $g_j(z) = \Sigma_n \zeta_{j,n} z^{\mu_j - n}$; $\zeta_{j,0} = \zeta_j$.

Extend analytically and compare:

$$f_i(z) = \Sigma_j c_{ij} g_j(z)$$

Goal: compute c_{ij}.

We'll show:

$$c_{ij} = e^{i\pi(\lambda_i - \mu_i)} \frac{\Pi_{k \neq i} \Gamma(\lambda_i - \lambda_k + 1) \Pi_{\ell \neq j} \Gamma(\mu_j - \mu_\ell)}{\Pi_{\ell \neq j} \Gamma(\lambda_i - \mu_\ell + 1) \Pi_{k \neq i} \Gamma(\mu_j - \lambda_k)}$$

Fact. The transport matrix depends only on the eigenvalues of P and $P - Q$. This delendence is holomoprhic.

Fact. $\sigma, \tau \in S_N$; $c_{ij}(\lambda_1, \ldots, \lambda_N, \mu_1, \ldots \mu_N) = c_{\sigma(i), \tau(j)}(\lambda_\sigma(1), \ldots, \lambda_\sigma(N), \mu_\tau(1), \ldots \mu_\tau(N))$.

Look at $\phi(f_1(z))$ because $\phi(f_1(z)) = \sum_j c_{ij} \phi(g_j(z))$. Recall power series for $f_1(z)$.

$\Sigma_{n \geq 0} (n + \lambda_1) \xi_{1,n} z^n = \Sigma_{n \geq 0} P \xi_{1,n} z^n + Q(1 + z + z^2 + \ldots) \Sigma_{n \geq 0} \xi_{1,n} z^n$.

$(n + \lambda_1 - P) \xi_{1,n} = Q(\xi_{1,0} + \cdots + \xi_{1,n-1})$

Let $\alpha_{1,n} = \phi(\xi_{1,0} + \cdots + \xi_{1,n})$.

By normalization $\alpha_{1,0} = 1$.

Black box (see Wasserman): $\alpha_{1,n} = \Pi_{j=1}^m \Pi_{m=1}^n \frac{m + \lambda_i - \mu_j}{m + \lambda_i - \lambda_j}$

Now get $\frac{\phi(f_1(z))}{z_1^\lambda (1-z)} = \Sigma_{n \geq 0} \alpha_{1,n} z^n = \Sigma_{n \geq 0} z^n \Pi_{j=1}^m \Pi_{m=1}^n \frac{m + \lambda_i - \mu_j}{m + \lambda_i - \lambda_j}$.

Restrict the λ_i's, μ_j's to real numbers; λ_i's, μ_j's are written in order $\lambda_i > \lambda_{i+1}$, $\lambda_1 + 1 > \mu_j > \lambda_j$ for all j.

Apply identity

$$\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} = \int_0^1 t^{a-1}(1-t)^{b-1}dt$$

(true when $a, b > 0$).

So,

$$\phi(f_1(z)) = (1-z)z^{\lambda_1}k\int\int\cdots\int(1-zt_2\cdots t_N)^{\mu_1-\lambda_1-a}\cdot\Pi_{j\neq 1}t_j^{\lambda_1-\mu_j}(1-t_j)^{\mu_j-\lambda_j-1}dt_j$$

with $k = \Pi_{j\neq 1}\frac{\Gamma(\lambda_1-\lambda_j+1)}{\Gamma(\lambda_1-\mu_j+1)\Gamma(\mu_j-\Lambda_j)}$

Black box: $\phi(g_j(z)) \approx |z|^{\mu_j}e^{\pi i\mu_j}$ is $z \in \mathbb{R}$ is large negative.

So $\phi(g_j(z)) \approx c_{11}|z|^{\mu_1}e^{\pi i\mu_1}$;

Thus $\phi(f_1(z)) \approx Ke^{i\pi\lambda_1}|z|^{\mu_1}\Pi_{j\neq 1}\in_0^1 t_j^{\mu_1-\mu_j+1}(1-t_j)^{\mu_j-\lambda_j-1}dt_j$

and undoing the β identity,

$$c_{11} = e^{i\pi(\lambda_1-\mu_1)}\Pi_{j\neq 1}\frac{\Gamma(\lambda_1-\Lambda_j+1)\Gamma(\mu_1-\mu_j)}{\Gamma(\lambda_1-\mu_j+1)\Gamma(\mu_1-\lambda_j)}$$