THE KNIZHNIK-ZAMOLODCHIKOV EQUATION

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Abstract. Notes from the "Conformal Field Theory and Operator Algebras workshop," August 2010, Oregon.

1. Motivation:

Why do we expect a differential equation to be useful in this formalism Wasserman's using?

Recall. λ is a tableau or "signature"; it gives rise to V_λ. If λ admissible for level ℓ, we get H_λ, an irrep of $\tilde{LG}_\ell \rtimes S^1_{rot^n}$. This has $H_\lambda(0) = V_\lambda$.

Heuristically, what are primary fields? $\mathrm{Hom}_G(V_\lambda \otimes W, V_\mu)$. Under favorable circumstances, get a map to $\mathrm{Hom}_{\tilde{LG} \rtimes S^1_{rot}}(H_\lambda \otimes C^\infty(S^1, W), H_\mu)$. Say this takes ϕ to φ.

Think of φ as $W^u \otimes \mathrm{Hom}^{unbd}(H_\lambda, H_\mu)$-value distribution.

Fourier modes: $\varphi(n) =$ "$\int_{S^1} \varphi(\zeta)\zeta^{-n}\frac{d\zeta}{2\pi\zeta}$" $= \varphi(\zeta^{-n})$.

This is not so bad; for $w \in W$, $\varphi(n)(w) L H_\lambda(k) \to H_\mu(k-n)$

Some notation: $\lambda_1, \lambda_2, \mu$ are tableau; W_1, W_2 are G-representations. Set $\mathcal{U} = \mathrm{Hom}_G(V_{\lambda_1} \otimes W_1 \otimes W_2, V_{\lambda_2})$.

$$\phi^1 : V_{\lambda_1} \otimes W_1 \to V_\mu$$
$$\phi^2 : V_\mu \otimes W_2 \to V_{\lambda_2}$$
$$\phi^2 \circ \phi^1 \in \mathcal{U}$$

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Available online at `http://math.mit.edu/~eep/CFTworkshop`. Please email `eep@math.mit.edu` with corrections and improvements!

replace ϕ^i by φ^i; consider $\underline{\varphi^2 \circ \varphi^1}$. We take (f, g) to $\varphi^2(f) \circ \varphi^1(g)$, for $f \in C^\infty(S^1, W_2)$, $g \in C^\infty(S^1, W_1)$, still ignoring issues about unbounded operators.

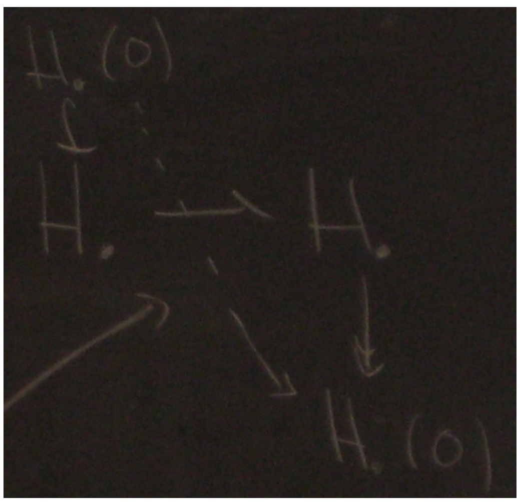
$\mathcal{U} \simeq \bigoplus_{\mu'} \mathrm{Hom}_G(V_{\mu'} \otimes W_1, V_{\lambda_2}) \otimes \mathrm{Hom}_G(V_{\lambda_1} \otimes W_2, V_{\mu'})$

With superscripts on φ being charges, subscripts being targets and sources.

$\phi^2 \circ \phi^1 = \Sigma_{\mu'}(?)\phi^{w_1}_{\lambda_2\mu'} \otimes \phi^{w_2}_{\mu'\lambda_1}$ gets sent to same think, with φs.

Play with power series: $f = \Sigma f_n z^n$, $g = \Sigma g_n w^n$.

$\varphi : f \mapsto \varphi(f)$.



$\underline{\varphi}$ defined to the the image under composition of the diagram above.

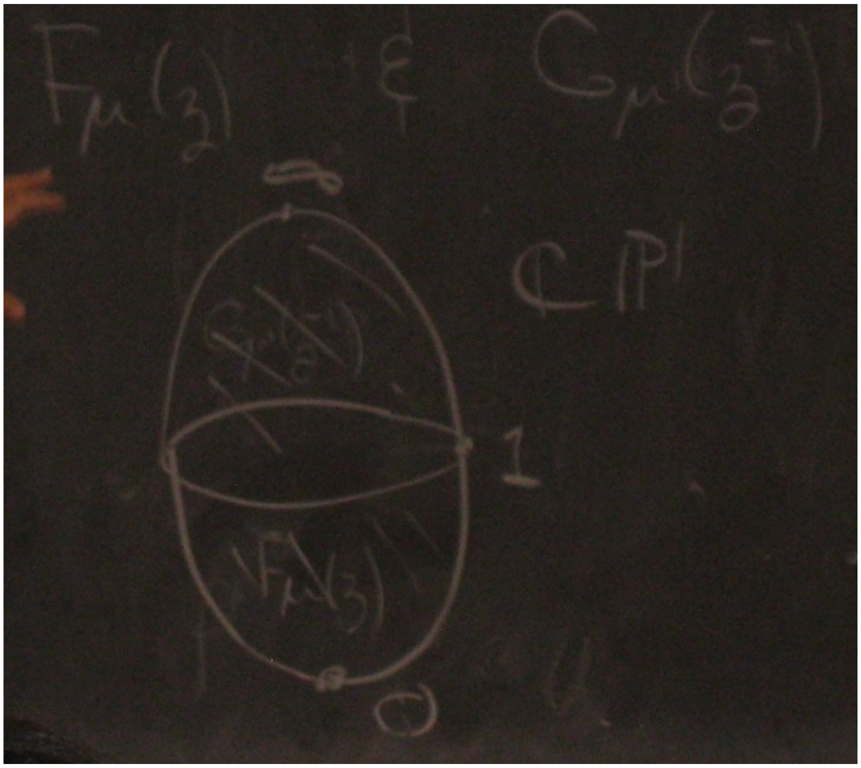
Four-point function: $F_\mu = \underline{\varphi^2(f) \circ \varphi^1(g)}$. By playing with power series, F_μ is a function of z/w with corefficients of the form $f_n g_{-n}$ times some number. ie, $F_\mu(\zeta) = \Sigma_{n \geq 0}\underline{\varphi^2(n) \circ \varphi^1(-n)}\zeta^n$.

Lemma 1.1. $\underline{\varphi^2(f)\varphi^1(g)} \in \mathcal{U}$ *and* $\underline{\varphi^2(f)\varphi^1(g)} = \int_{S^1-\{1\}} (\tilde{f} \star g)(\zeta) F_\mu(\zeta) \frac{d\zeta}{2\pi\zeta}$, *where $\tilde{f}$ is f 'backwards' around the circle. This is assuming f and g have disjoint support, and F_μ extends continuously to $S^1 - \{1\}$.*

It's easy to check

$(\tilde{(f)} \star g)(\zeta) = (\tilde{(g)} \star f)(\zeta^{-1})$;

This means we have relations between holomorphic functions $F_\mu(\zeta)$ and $G_\mu(\zeta^{-1})$;



And, relations on these things that are called fourpoint functions (F_μ) give relations between primary fields.

Question: what's a four point function? why's it called that? What's it do?

Answer: So, given $f, g \in C^\infty(S^1, \mathbb{C})$ and $w_1, w_2 \in W_1, W_2$, we have $\underline{\varphi^2(w_2 f) \circ \varphi^1(w_1 g)}$.

underline here means to restrict to vectors v_λ in $H_\lambda(0)$; do this projection by taking inner products

$\langle \underline{\varphi^2(w_2 f) \circ \varphi^1(w_1 g)} v_{\lambda_1}, v_{\lambda_2}, \rangle$ – four inputs, hence 'four point function.'

Motivation: What do differential equations have to do with anything?

Answer: transport coefficients?

2. Generalities

Wanted "nice" OCE setisfied by F_μ and $G_\mu(\zeta^{-1})$. Wasserman's "Basic ODE:"

$$\frac{df}{dz} = (\frac{P}{z} + \frac{Q}{1-z})f$$

f is $\mathcal{U}$-valued funtion, $P, Q \in \mathrm{End}(\mathcal{U})$.

This equations is linear, first order, with regular singularities.

Example. $\mathcal{U} = \mathbb{C}$.

$$\frac{df}{f} = \frac{dz}{z}(P + Q\frac{z}{1-z})$$

here $\frac{df}{f} = d\log f$, $\frac{dz}{z} = d\log z$, and $P + Q\frac{z}{1-z}$ is holomorphic near 0.

This gives us $f = \zeta^P$ – a function homomorphic near 0 and unique up to scalars.

Note. If P has eigenvalues not differing by integers, then solutions near 0 look like $f = \Sigma a_i \tilde{f}(\zeta)\xi_i\zeta$

Canonical basis of U corresponding to eigenvalues of P, and corresponding to solutions.

General fact: Such a "nice" differential equation of $\mathbb{C}P^1$ with regular singularities at 0, 1, ∞ correspond to local systems on $\mathbb{C}P^1 - \{0, 1, \infty\}$ (with variables $\mathcal{U}$). These correspond to maps $\pi_1(\mathbb{C}P^1 - \{0, 1, \infty\}) \to GL(\mathcal{U})$.

3. Motivation

Why do we get a differential equation?

Suppose we're given $n + 1$ points in $\mathbb{C}P^1$ and coordinates $D \to \mathbb{C}P^1$; this gives us a vectors space non-canonical $\mathcal{U}$.

A change of coordinates gives us an isomorphism of vectors spaces.

This sort of data gives us a variable on $M_{0,n+1}$ – the moduli space of $(n+1)$ distinct ordered points in $\mathbb{C}P^1$.

A change of coordinates not necessarily preserving 0 also gives an isomorphism of vector spaces.

Think of this giving us a parallel transport corresponding to some connection on this vector bundle.

So, "nice" differential equations means something coming from a flat connection.

$n = 3$ (4 points). $M_{0,4} \simeq \mathbb{C}P^1 - \{0, 1, \infty\}$.

The KZ equation in Wasserman is this construction on $M_{0,4}$.

Other KZ equations in the literature: consider the configuration space of n points in $\mathbb{C}$, $\mathbb{C}^n - \Delta$. Take $\cup\{\infty\}$ to get $M_{0,n+1}$.

The setup for KZ: $G = SU(N)$; $\mathfrak{g}$; X_k is orthonormal basis for $\mathfrak{g}$.

The *Casimir* Ω (not to be confused with vacuum!) is $\Sigma X_k \otimes X_k$.

Lemma 3.1. *Ω is in the center of $\mathcal{U}(\mathfrak{g})$ (universal enveloping algebra).*

Δ_λ is the scalar by which Ω acts of V_λ.

Let $\mathcal{U} = \mathrm{Hom}(V_{\lambda_1} \otimes W_1 \otimes W_2 \otimes \cdot \otimes W_{n-1}, V_{\lambda_2})$

$\mathfrak{g}$ acts on each vector space inside there.

$\Omega_{i,j} = -\Sigma \pi_i(X_k) \otimes \pi_j(X_k) \in \mathrm{End}(\mathcal{U})$; the Casimir element acting separately on the i, j components.

Theorem 3.2. *F_μ satisfies*

$$(N + \ell)\frac{dF_\mu}{dz} = (\frac{\Omega_{23} - (\Delta_\mu - \Delta_{w_1} - \Delta_{w_2})/2}{z} + \frac{\Omega_{12}}{z - 1})F$$

Theorem 3.3. *$f_\mu = \zeta^{(\Delta_\mu - \Delta_{w_1} - \Delta_{w_2})/(2(N+\ell))} F_\mu(\zeta)$ satisfies $(N + \ell)\frac{df_\mu}{dz} = (\frac{\Omega_{23}}{\zeta} + \frac{\Omega 12}{\zeta - 1})f_\mu$*