PLANAR ALGEBRAS & KNOTS

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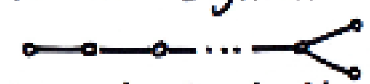
SUMMARY

Planar algebras describe the structure which knots and subfactors of von Neumann algebras share.

They explain the connection between knots and the Temperley-Lieb algebra which led to the discovery of the Jones polynomial.

We describe the planar algebra associated to the Dynkin diagram

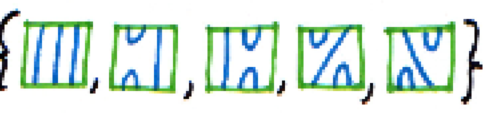
D_{2n} = 

and applications to knot theory.

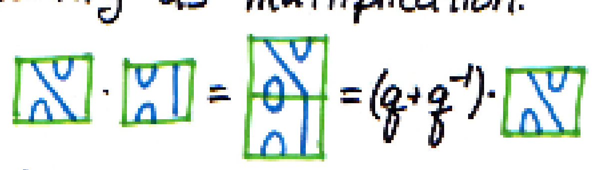
EXAMPLES

TEMPERLEY-LIEB

Def'n: TL_n is the span (over $\mathbb{C}$) of non-crossing pairings of $2n$ points.

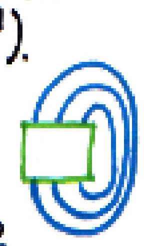
$TL_3 = \mathrm{Span}_{\mathbb{C}}\{$  $\}$

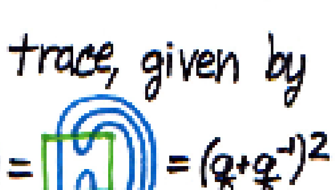
- TL_n is an algebra, with stacking as multiplication.

EG: 

$= (q+q^{-1})\cdot$

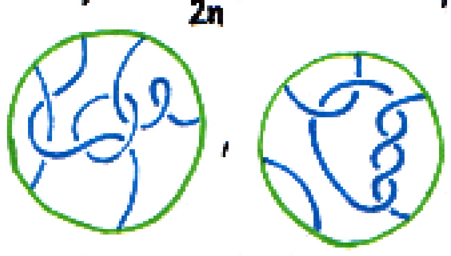
If loops appear, throw them out and multiply by $(q+q^{-1})$.

- TL_n has a trace, given by 

EG: $tr($  $) = (q+q^{-1})^2$

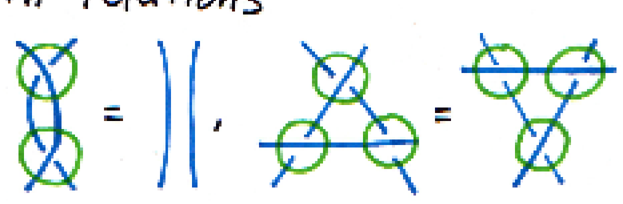
FRAMED LINKS

Def'n: FL_n is the span of arrangements of string in a disk, with $2n$ fixed endpoints.



$\in FL_3$

- FL_n is generated (as a planar algebra) by ⊗, with relations



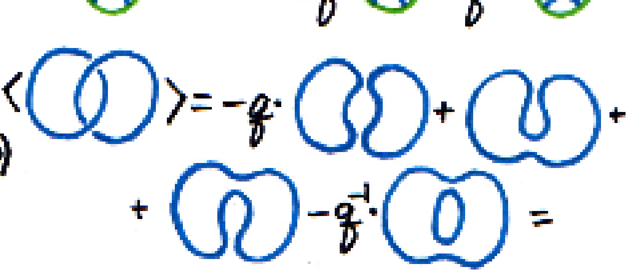
THE JONES POLYNOMIAL

Def'n: The Kauffman bracket is a homomorphism of planar algebras:

$\langle\cdot\rangle : FL_n \longrightarrow TL_n$

⊗ $\longmapsto i q^{1/2}$ ⦶ $- i q^{-1/2}$ ⦵

EG: (n=0)



$= -q^3 - q - q^{-1} - q^{-3}$

Def'n: The Jones polynomial

$J(L) = (iq^{3/2})^{-\mathrm{writhe}(L)} \cdot \langle L \rangle$

is a link invariant

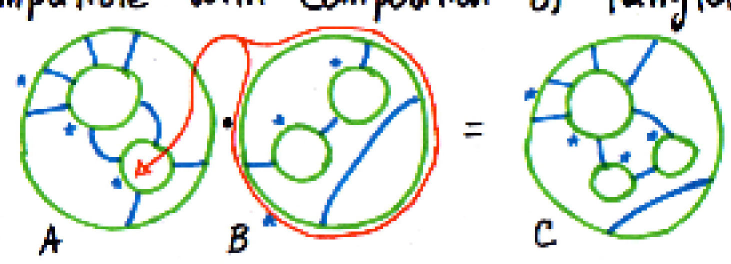
PLANAR ALGEBRAS

Def'n A planar algebra is

- A tower of vector spaces $V_0 \subset V_1 \subset V_2 \subset \cdots$
- An action by planar tangles

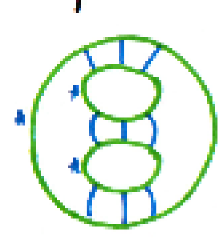
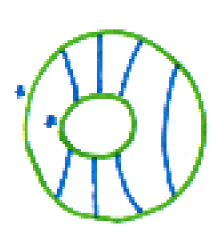
eg.  $: V_1 \otimes V_2 \otimes V_1 \longrightarrow V_3$

compatible with composition of tangles:



$A(B(v_1, v_2), w) = C(v_1, v_2, w)$

Some Special Tangles

- Multiplication  $: V_k \otimes V_k \longrightarrow V_k$
- Trace  $: V_k \longrightarrow V_0$
- Inclusion  $: V_k \longrightarrow V_{k+1}$

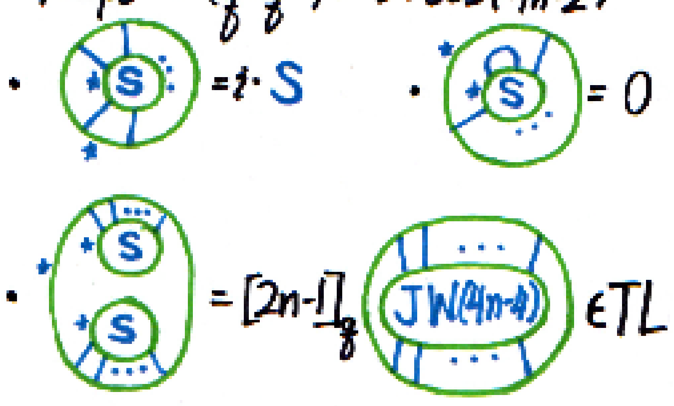
NEW WORK

D_{2n}

Def'n: The D_{2n} planar algebra is generated (over planar tangles) by

with relations

- loops $= (q+q^{-1}) = 2\cos(\frac{\pi}{4n-2})$

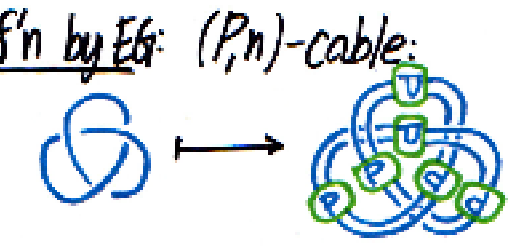


D_{2n} is "almost braided:"

D_{2n} LINK INVARIANT

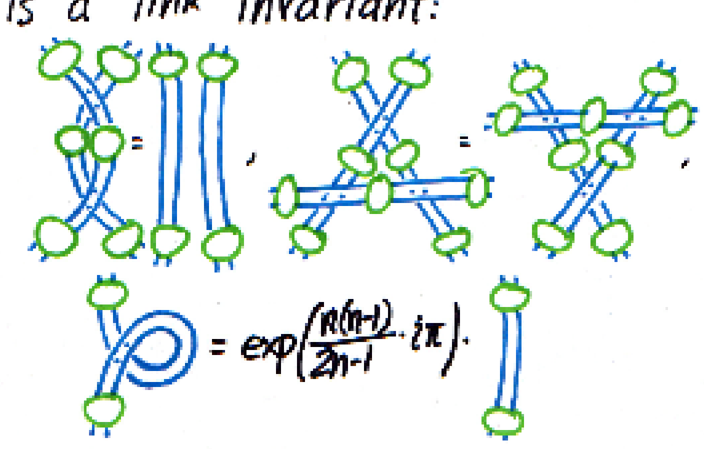
- $P = \frac{1}{2}(S + JW(2n-2))$ is a projection

Def'n by EG: (P,n)-cable:



Def'n: $J_{P,n}(L) := \left(\exp\left(\frac{n(n-1)}{2n-1}\cdot i\pi\right)\right)^{-\mathrm{writhe}(L)} \cdot \langle (P,n)\text{-cable}(L) \rangle$

is a link invariant:



KNOT COINCIDENCES

- $J_{P,2}(K) = \frac{1}{2} CJ_2(K)|_{q=\exp(2\pi i/12)} = 1$
- $J_{P,3}(K) = \frac{1}{2} CJ_3(K)|_{q=\exp(2\pi i/20)} = J(K)|_{q=\exp(-2\pi i/10)}$
- $J_{P,4}(K) = \frac{1}{2} CJ_4(K)|_{q=\exp(2\pi i/28)} = 2\,\mathrm{HOMFLYPT}(K)|_{\exp(2\pi i \cdot 3/14),\ i \cdot 2\sin(2\pi i/14)}$
- $J_{P,5}(K)$ relates CJ_5 and the Kauffman polynomial.
- $J_{P,7}(K)$ relates CJ_7 and the quantum G_2 invariant.

REFERENCES

1. VFR Jones, Planar Algebras, I, arXiv:math.QA/9909027
2. L Kauffman, State models and the Jones polynomial, Topology 26 (1987), 395–407
3. S Morrison, E Peters & N Snyder, Skein theory for the D_{2n} planar algebra, arXiv:0808.0764
4. S Morrison, E Peters & N Snyder, Knot polynomial identities. Will appear at math.berkeley.edu/~eep